

# Chains conditions in algebraic lattices

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Le texte comprend deux résumés -l'un en anglais, l'autre en français; une introduction en français, suivie du corps de la thèse formée de quatre chapitres en anglais.

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## Abstract

This work studies the relationship between the chains of an algebraic lattice and the order structure of the join-semilattice of its compact elements. The results are presented into four chapters, each corresponding to a paper written in collaboration with Maurice Pouzet.

- (1) A characterization of well-founded algebraic lattices, 19p.  
arXiv:0812.2300.
- (2) The length of chains in algebraic lattices, Les annales ROAD du LAID3, special issue 2008, pp 379-390 (proceedings of ISOR'08, Algiers, Algeria, Nov 2-6, 2008).
- (3) The length of chains in algebraic modular lattices, Order, 24(2007) 224-247.
- (4) Infinite independent sets in distributive lattices, Algebra Universalis, 53(2005) 211-225.

Our first studies on this theme have appeared in our doctoral thesis [6] presented in Lyon in 1992. A part of our results is included in Chapter 3 and 4 of the present work.

Here are our main results (the necessary definitions can be found in the last section of this volume).

We show that for every order type  $\alpha$  there is a list  $\mathbb{B}_\alpha$  of join-semilattices, with cardinality at most  $2^{|\alpha|}$ , such that an algebraic lattice  $L$  contains a chain of order type  $I(\alpha)$  if and only if the join-subsemilattice  $K(L)$  of its compact elements contains a join-semilattice isomorphic to a member of  $\mathbb{B}_\alpha$ . (Theorem 2.3, Chapter 2).

We conjecture that when  $\alpha$  is countably infinite, there is a finite list. The following result supports this conjecture: Let  $[\omega]^{<\omega}$  be the set of finite subsets of  $\omega$  ordered by inclusion. Then, among the join-subsemilattices of  $[\omega]^{<\omega}$  belonging to  $\mathbb{B}_\alpha$ , one embeds in all others as a join-semilattice (cf. Theorem 2.4, Chapter 2).

We also show that an algebraic lattice  $L$  is well-founded if and only if the join-semilattice  $K(L)$  of compact elements  $L$  is well-founded and contains no join-semilattice isomorphic to  $\underline{\Omega}(\omega^*)$  or to  $[\omega]^{<\omega}$  (Theorem 1.2, Chapter 1).

We describe the countable indivisible order types  $\alpha$  such that: for every modular algebraic lattice  $L$ ,  $L$  contains no chain of order type  $\alpha$  if and only if the join-semilattice of its compact elements contains neither  $\alpha$  nor a join-semilattice isomorphic to  $[\omega]^{<\omega}$  (Theorem 3.1, Chapter 3). The chains  $\omega^*$  et  $\eta$  are among these order types  $\alpha$ .

We identify two meet-semilattices  $\Gamma$  et  $\Delta$ . We show (Theorem 4.3, Chapter 4) that: *for every distributive lattice  $T$ , the following properties are equivalent:*

- (i)  *$T$  contains a join-subsemilattice isomorphic to  $[\omega]^{<\omega}$ ;*
- (ii) *The lattice  $[\omega]^{<\omega}$  is a quotient of a sublattice of  $T$ ;*
- (iii)  *$T$  contains a sublattice isomorphic to  $I_{<\omega}(\Gamma)$  ou à  $I_{<\omega}(\Delta)$ .*

This abstract is followed by its french version and by an introduction in french. The four chapters are in english.



## Résumé

Cette thèse porte sur le rapport entre la longueur des chaînes d'un treillis algébrique et la structure d'ordre du sup-treillis de ses éléments compacts. Les résultats obtenus ont donné lieu à quatre articles écrits en collaboration avec Maurice Pouzet et constituant chacun un chapitre de la thèse:

- (1) A characterization of well-founded algebraic lattices, 19p.  
arXiv:0812.2300.
- (2) The length of chains in algebraic lattices, Les annales ROAD du LAID3, special issue 2008, pp 379-390 (proceedings of ISOR'08, Algiers, Algeria, Nov 2-6, 2008).
- (3) The length of chains in algebraic modular lattices, Order, 24(2007) 224-247.
- (4) Infinite independent sets in distributive lattices, Algebra Universalis, 53(2005) 211-225.

Nos premières recherches sur ce thème sont apparues dans notre thèse de doctorat [6] Lyon 1992. Une partie d'entre elles est incluse dans les chapitres 3 et 4 du présent travail. Voici nos principaux résultats (on pourra trouver en appendice les définitions nécessaires).

Nous montrons que: *si  $\alpha$  est un type d'ordre, il existe une liste de sup-treillis, soit  $\mathbb{B}_\alpha$ , de taille au plus  $2^{|\alpha|}$ , telle que quel que soit le treillis algébrique  $L$ ,  $L$  contient une chaîne de type  $I(\alpha)$  si et seulement si le sous sup-treillis  $K(L)$  de ses éléments compacts contient un sous sup-treillis isomorphe à un membre de  $\mathbb{B}_\alpha$ .* (Theorem 2.3, Chapter 2).

Nous conjecturons que si  $\alpha$  est dénombrable il existe une liste finie.

Considérant l'ensemble  $[\omega]^{<\omega}$  formé des parties finies de  $\omega$  et ordonné par inclusion, nous montrons à l'appui de cette conjecture que: *parmi les sous sup-treillis de  $[\omega]^{<\omega}$  appartenant à  $\mathbb{B}_\alpha$ , l'un d'eux se plonge comme sous sup-treillis dans tous les autres* (cf. Theorem 2.4, Chapter 2).

Comme résultat positif, nous montrons que: *un treillis algébrique  $L$  est bien fondé si et seulement si le sup-treillis  $K(L)$  des éléments compacts de  $L$  est bien fondé et ne contient pas de sous sup-treillis isomorphe à  $\underline{\Omega}(\omega^*)$  ou à  $[\omega]^{<\omega}$*  (Theorem 1.2, Chapter 1).

Nous décrivons les types d'ordre indivisibles dénombrables  $\alpha$  tels que: *quel que soit le treillis algébrique modulaire  $L$ ,  $L$  ne contient pas de chaîne de type  $\alpha$  si et seulement si le sup-treillis de ses éléments compacts ne contient ni  $\alpha$  ni de sous sup-treillis isomorphe à  $[\omega]^{<\omega}$*  (Theorem 3.1, Chapter 3). Parmi eux figurent  $\omega^*$  et  $\eta$ .

Nous identifions deux inf-treillis  $\Gamma$  et  $\Delta$ . Nous montrons (Theorem 4.3, Chapter 4) que: *pour un treillis distributif  $T$ , les propriétés suivantes sont équivalentes:*

- (i)  *$T$  contient un sous sup-treillis isomorphe à  $[\omega]^{<\omega}$ ;*
- (ii) *Le treillis  $[\omega]^{<\omega}$  est quotient d'un sous-treillis de  $T$ ;*
- (iii)  *$T$  contient un sous-treillis isomorphe à  $I_{<\omega}(\Gamma)$  ou à  $I_{<\omega}(\Delta)$ .*

## Introduction

La notion d'opérateur de fermeture est centrale en mathématiques. Elle est également un outil de modélisation dans plusieurs sciences appliquées comme l'informatique et les sciences sociales (eg. logique et modélisation du calcul, bases de données relationnelles, fouille de données...). Dans beaucoup d'exemples concrets, les opérateurs de fermeture satisfont une propriété d'engendrement fini, et on dit que ceux-ci sont algébriques. À un opérateur de fermeture est associé un treillis complet, le treillis de ses fermés. Et ce treillis est algébrique si et seulement si la fermeture est algébrique. Du fait de l'importance de ces opérateurs de fermeture, les treillis algébriques sont des objets privilégiés de la théorie des treillis.

Un exemple de base est l'ensemble, ordonné par inclusion, des sections initiales d'un ensemble ordonné (ou préordonné). C'est celui qui motive notre recherche. Son importance vient, pour nous, de la théorie des relations et de la logique. Les structures relationnelles étant préordonnées par abritement, les sections initiales correspondent aux classes de modèles finis des théories universelles tandis que les idéaux sont les âges de structures relationnelles introduits par Fraïssé [13]. Et toute une approche de l'étude des structures relationnelles finies peut être exprimée en termes d'ordre et tout particulièrement en termes de ce treillis des sections initiales [30, 31, 32, 40].

Notre travail porte sur la façon dont les chaînes d'un treillis algébrique  $L$  se reflètent dans la structure du sup-treillis  $K(L)$  constitué des éléments compacts de ce treillis. L'objectif étant de déterminer les sup-treillis que  $K(L)$  doit contenir pour assurer que  $L$  contienne une chaîne d'un type donné  $\alpha$ .

Pour la compréhension de ce qui suit, rappelons qu'un treillis  $L$  est *algébrique* s'il est complet et si tout élément est suprémum d'éléments compacts. L'ensemble  $K(L)$  des éléments compacts de  $L$  est un *sup-treillis* (c'est à dire un ensemble ordonné dans lequel toute paire d'éléments a un suprémum) admettant un plus petit élément et, de plus, l'ensemble  $J(K(L))$  des idéaux de  $K(L)$  est un treillis complet isomorphe à  $L$ . Réciproquement, si  $P$  est un sup-treillis ayant un plus petit élément, l'ensemble  $J(P)$  des idéaux de  $P$  est un treillis algébrique dont l'ensemble des éléments compacts est isomorphe à  $P$ . Ainsi, pour revenir à l'exemple qui motive notre recherche, si  $L$  est l'ensemble des sections initiales d'un ensemble ordonné  $Q$ , alors  $K(L)$  est égal à l'ensemble  $I_{<\omega}(Q)$  des sections initiales de  $Q$  qui sont finiment engendrées. Et  $I(Q)$  est isomorphe à  $J(I_{<\omega}(Q))$ .

Exprimé en termes de sup-treillis et d'idéaux, notre travail porte donc sur le rapport entre la longueur des chaînes d'idéaux d'un sup-treillis admettant un plus petit élément et la structure d'ordre de ce sup-treillis.

Il s'inspire d'un rapport entre longueur des chaînes d'idéaux d'un ensemble ordonné  $P$  quelconque et structure d'ordre de cet ensemble ordonné constaté par Pouzet et Zaguia, 1985 [34].

Étant donnée une chaîne de type  $\alpha$ , désignons par  $I(\alpha)$  le type d'ordre de l'ensemble des sections initiales de cette chaîne. Disons que  $\alpha$  est *indécomposable* si la chaîne s'abrite dans chacune de ses sections finales non-vides.

Pouzet et Zaguia obtiennent le résultat suivant (cf. [34] Theorem 4 pp. 162 et Theorem 2.2, Chapter 2).

**Théorème 0.1.** *Soit  $\alpha$  un type d'ordre indécomposable dénombrable.*

*Il existe une liste finie  $A_1^\alpha, \dots, A_{n_\alpha}^\alpha$  d'ensembles ordonnés, tels que pour tout ensemble ordonné  $P$ , l'ensemble  $J(P)$  des idéaux de  $P$  ne contient pas de chaîne de type  $I(\alpha)$  si et seulement si  $P$  ne contient aucun sous-ensemble isomorphe à un des  $A_1^\alpha, \dots, A_{n_\alpha}^\alpha$ .*

Les  $A_1^\alpha, \dots, A_{n_\alpha}^\alpha$  sont les "obstructions" à l'existence d'une chaîne d'idéaux de type  $I(\alpha)$ . Une obstruction typique s'obtient par *sierpinskisation monotone*: on se donne une bijection  $\varphi$  des entiers sur une chaîne, dont le type  $\omega\alpha$  est la somme de  $\alpha$  copies de la chaîne  $\omega$  des entiers, telle que  $\varphi^{-1}$  soit croissante sur chaque  $\omega.\{\beta\}$  pour  $\beta \in \alpha$ . On ordonne  $\mathbb{N}$  en posant  $x \leq y$  si  $x \leq y$  dans l'ordre naturel et  $\varphi(x) \leq \varphi(y)$  dans l'ordre de  $\omega\alpha$ . On constate que l'ensemble ordonné obtenu, augmenté d'un plus petit élément, s'il n'en a pas, contient une chaîne d'idéaux de type  $I(\alpha)$ . En outre, tous les ensembles ordonnés obtenus par ce procédé se plongent les uns dans les autres; de ce fait, on les désigne par le même symbole  $\underline{\Omega}(\alpha)$  (cf. Lemma 3.4.3 pp. 167 [34]). Si  $\alpha$  n'est ni  $\omega$ , ni  $\omega^*$ , le dual de  $\omega$  et ni  $\eta$  (le type d'ordre des rationnels), les autres obstructions s'obtiennent au moyen de sommes lexicographiques d'obstructions correspondant à des chaînes de type strictement inférieur à  $\alpha$ .

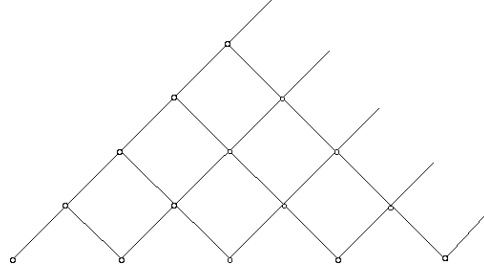
Il est naturel de se demander ce que devient le résultat ci-dessus si, au lieu d'ensembles ordonnés, on considère des sup-treillis. Cette question est au coeur de notre travail.

Afin de raccourcir la suite de l'exposé, formalisons un peu:

Désignons par  $\mathbb{E}$  la classe des ensembles ordonnés. Étant donnés  $P, P' \in \mathbb{E}$  disons que  $P$  s'abrite dans  $P'$  et notons  $P \leq P'$  si  $P$  est isomorphe à une partie de  $P'$ . Cette relation est un préordre. Deux ensembles ordonnés  $P$  et  $P'$  tels que  $P \leq P'$  et  $P' \leq P$  sont dits *équimorphes*. Pour un type d'ordre  $\alpha$ , désignons par  $\mathbb{E}_{-\alpha}$  la classe formée des éléments  $P \in \mathbb{E}$  tels que  $J(P)$  n'abrite pas de chaîne de type  $I(\alpha)$ . Pour une partie  $\mathbb{B}$  de  $\mathbb{E}$  désignons par  $\uparrow \mathbb{B}$  la classe des  $P \in \mathbb{E}$  abritant un élément de  $\mathbb{B}$  et posons  $Forb(\mathbb{B}) := \mathbb{E} \setminus \uparrow \mathbb{B}$  ("Forb" pour "forbidden"). Le résultat ci-dessus s'écrit

$$(1) \quad \mathbb{E}_{-\alpha} = Forb(\{A_1^\alpha, \dots, A_{n_\alpha}^\alpha\}).$$

Remplaçons la classe  $\mathbb{E}$  par la classe  $\mathbb{J}$  des sup-treillis  $P$  admettant un plus petit élément. Pour un type d'ordre  $\alpha$ , désignons par  $\mathbb{J}_\alpha$  la classe des  $P$  dans  $\mathbb{J}$  tels que  $J(P)$  abrite une chaîne de type  $I(\alpha)$  et posons  $\mathbb{J}_{-\alpha} := \mathbb{J} \setminus \mathbb{J}_\alpha$ . Pour une partie  $\mathbb{B}$  de  $\mathbb{J}$ , désignons par  $\uparrow \mathbb{B}$  la classe des  $P$  dans  $\mathbb{J}$  contenant un sous sup-treillis (et non un

FIGURE 0.1.  $\Omega(\omega^*)$ 

sous-ensemble ordonné) isomorphe à un membre de  $\mathbb{B}$  et posons  $Forb_{\mathbb{J}}(\mathbb{B}) := \mathbb{J} \setminus \uparrow \mathbb{B}$ .

Nous pouvons préciser notre question ainsi:

QUESTIONS 0.1. (1) Pour un type d'ordre  $\alpha$ , quelles sont les parties  $\mathbb{B}$  de  $\mathbb{J}$  les plus simples possibles telles que:

(2)  $\mathbb{J}_{-\alpha} = Forb_{\mathbb{J}}(\mathbb{B})$ .

(2) Est ce que pour toute chaîne dénombrable  $\alpha$  on peut trouver  $\mathbb{B}$  fini tel que  $\mathbb{J}_{-\alpha} = Forb_{\mathbb{J}}(\mathbb{B})$ ?

Notre travail est consacré à ces questions. Nous n'y répondons que partiellement.

Pour répondre à la deuxième question, il est tentant d'utiliser le Théorème 0.1. En effet, si l'ensemble  $J(P)$  des idéaux d'un sup-treillis  $P$  contient une chaîne de type  $I(\alpha)$  alors  $P$  doit contenir, comme ensemble ordonné, un des  $A_i^\alpha$ . Et par conséquent,  $P$  doit contenir, comme sous sup-treillis, le sup-treillis engendré par  $A_i^\alpha$  dans  $P$ . Mais nous ne savons pas décrire les sup-treillis que l'on peut engendrer au moyen des  $A_i^\alpha$ . Notre approche consiste à adapter, quand cela paraît possible, la preuve du Théorème 0.1. Nous présentons ci-dessous les résultats obtenus.

## 1. Résultats généraux

Notons immédiatement que si  $\alpha$  est le type d'une chaîne finie on peut prendre  $\mathbb{B} = \{1 + \alpha\}$ . Ceci est encore vrai si  $\alpha$  est la chaîne  $\omega$  des entiers. Un cas plus intéressant est celui de la chaîne  $\omega^*$  des entiers négatifs. Les ensembles ordonnés n'abritant pas la chaîne  $\omega^*$  sont dits *bien fondés*; ensembles ordonnés dont toute partie non vide contient un élément minimal, ils servent à modéliser le raisonnement par induction.

Notons  $\Omega(\omega^*)$  l'ensemble  $[\omega]^2$  des parties à deux éléments de  $\omega$ , qu'on identifie aux paires  $(i, j)$ ,  $i < j < \omega$ , muni de l'ordre suivant:  $(i, j) \leq (i', j')$  si et seulement si  $i' \leq i$  et  $j \leq j'$ . Soit  $\underline{\Omega}(\omega^*) := \Omega(\omega^*) \cup \{\emptyset\}$  obtenu en ajoutant un plus petit élément. Notons  $[\omega]^{<\omega}$  l'ensemble, ordonné par inclusion, des parties finies de  $\omega$ .

Les ensembles  $\underline{\Omega}(\omega^*)$  et  $[\omega]^{<\omega}$  sont des treillis bien-fondés, tandis que les treillis algébriques  $J(\underline{\Omega}(\omega^*))$  et  $J([\omega]^{<\omega})$  ne le sont pas (par exemple  $J([\omega]^{<\omega})$  est

isomorphe à  $\mathfrak{P}(\omega)$ , l'ensemble des parties de  $\omega$ ). Ces deux ensembles sont incontournables, en effet:

$$(3) \quad \mathbb{J}_{-\omega^*} = \text{Forb}_{\mathbb{J}}(\{1 + \omega^*, \underline{\Omega}(\omega^*), [\omega]^{<\omega}\}).$$

C'est le premier résultat significatif de ce travail. Il se reformule de façon plus accessible comme suit:

**Théorème 0.2.** (cf. Theorem 1.2, Chapter 1) *Un treillis algébrique  $L$  est bien fondé si et seulement si le sup-treillis  $K(L)$  des éléments compacts de  $L$  est bien fondé et ne contient pas de sous sup-treillis isomorphe à  $\underline{\Omega}(\omega^*)$  ou à  $[\omega]^{<\omega}$ .*

Notons que la liste donnée par le Théorème 0.1 dans le cas  $\alpha = \omega^*$  a un terme de moins. On a en effet (cf. [34] Theorem 1. pp.160):

$$(4) \quad \mathbb{E}_{-\omega^*} = \text{Forb}\{\omega^*, \underline{\Omega}(\omega^*)\}$$

Ceci tient au fait que  $\underline{\Omega}(\omega^*)$  est isomorphe comme ensemble ordonné à un sous-ensemble de  $[\omega]^{<\omega}$ , mais non pas comme sup-treillis.

Après les chaînes finies, les chaînes  $\omega$  et  $\omega^*$ , une chaîne typique est la chaîne  $\eta$  des rationnels. Les ensembles ordonnés n'abritant pas la chaîne des rationnels sont dits *dispersés*.

Notons  $\underline{\Omega}(\eta)$  l'ensemble des couples d'entiers  $(n, \frac{i}{2^n})$  tels que  $0 \leq i < 2^n$  ordonné de sorte que:  $(n, \frac{i}{2^n}) \leq (m, \frac{j}{2^m})$  si  $n \leq m$  et  $\frac{i}{2^n} \leq \frac{j}{2^m}$ . Ainsi  $\underline{\Omega}(\eta)$  est un sous-ensemble du produit de la chaîne  $\omega$  et de la chaîne  $D$  des nombres dyadiques de l'intervalle  $[0, 1[$ . C'est en fait un sous sup-treillis de ce produit.

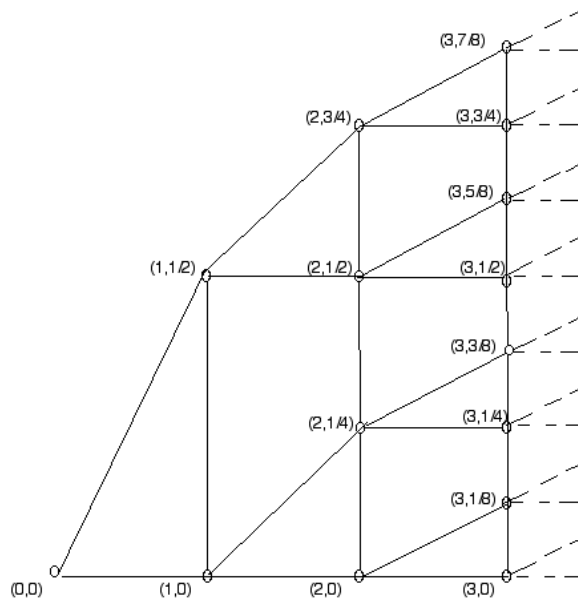


FIGURE 0.2.  $\underline{\Omega}(\eta)$

Dans [34] (Theorem 2. pp.161) il est prouvé que:

$$(5) \quad \mathbb{E}_{-\eta} = \text{Forb}\{\eta, \underline{\Omega}(\eta)\}$$

Nous ne savons pas répondre à la question suivante:

**Question 0.2.** *Est-ce que  $\mathbb{J}_{-\eta} = \text{Forb}_{\mathbb{J}}(\{1 + \eta, [\omega]^{<\omega}, \underline{\Omega}(\eta)\})$ ?*

Si  $\alpha$  est un type d'ordre arbitraire, nous obtenons le résultat suivant (analogue à celui déjà obtenu en [34], cf. Theorem 3 pp.162).

**Théorème 0.3.** *(cf. Theorem 2.3, Chapter 2) Soit  $\alpha$  un type d'ordre. Il existe une liste de sup-treillis, soit  $\mathbb{B}_{\alpha}$ , de taille au plus  $2^{|\alpha|}$ , telle que pour tout sup-treillis  $P$ , l'ensemble  $J(P)$  des idéaux de  $P$  contient une chaîne de type  $I(\alpha)$  si et seulement si  $P$  contient un sup-treillis isomorphe à un membre de  $\mathbb{B}_{\alpha}$ .*

C'est très loin d'une réponse positive à la seconde de nos questions, à savoir l'existence d'une liste finie lorsque  $\alpha$  est dénombrable.

Si cette question a une réponse positive alors il y a une seule liste de cardinal minimum, pourvu que des sup-treillis  $P, P'$  qui se plongent l'un dans l'autre comme sup-treillis soient identifiés. En effet, préordonnons  $\mathbb{J}$  en posant  $P \leq P'$  si  $P$  se plonge dans  $P'$  comme sous sup-treillis. Dans l'ordre quotient,  $\mathbb{J}_{\alpha}$  est une section finale. Et si  $\mathbb{B}$  est une liste finie de cardinal minimum alors dans ce quotient c'est l'ensemble des éléments minimaux de  $\mathbb{J}_{\alpha}$ .

En fait notre seconde question équivaut à:

**Question 0.3.** (1) *Est-ce que  $\mathbb{J}_{\alpha}$  admet un nombre fini d'éléments minimaux?*

(2) *Est-ce que tout élément de  $\mathbb{J}_{\alpha}$  majore un élément minimal?*

Par exemple  $1 + \alpha \in \mathbb{J}_{\alpha}$ . En effet  $J(1 + \alpha) = 1 + J(\alpha) = I(\alpha)$ . De plus  $1 + \alpha$  est minimal dans  $\mathbb{J}_{\alpha}$ . En effet, soit  $Q \in \mathbb{J}$  tel que  $Q \leq 1 + \alpha$ , on a  $Q = 1 + \beta$ . Comme  $J(Q) = I(\beta)$ , si  $Q \in \mathbb{J}_{\alpha}$ ,  $I(\alpha) \leq I(\beta)$ . Ce qui implique  $\alpha \leq \beta$  et  $1 + \alpha \leq Q$ .

Lorsque  $\alpha$  est dénombrable, le treillis  $[\omega]^{<\omega}$  joue un rôle central. Nous avons:

$$(6) \quad [\omega]^{<\omega} \in \mathbb{J}_{\alpha}.$$

En effet  $J([\omega]^{<\omega})$  est isomorphe à  $\mathfrak{P}(\omega)$  lequel contient une chaîne de type  $I(\alpha)$  pour tout  $\alpha$  dénombrable.

Nous montrons que vis à vis du préordre sur  $\mathbb{J}$ , l'ensemble des sous sup-treillis de  $[\omega]^{<\omega}$  qui appartiennent à  $\mathbb{J}_{\alpha}$  a un plus petit élément:

**Théorème 0.4.** *(cf. Theorem 2.4, Chapter 2) Pour tout  $\alpha$  dénombrable,  $\mathbb{J}_{\alpha}$  contient un sup-treillis  $Q_{\alpha}$  qui se plonge comme sous sup-treillis dans tout sous sup-treillis de  $[\omega]^{<\omega}$  appartenant à  $\mathbb{J}_{\alpha}$ . Ce treillis  $Q_{\alpha}$  étant égal à:*

- $1 + \alpha$  si  $\alpha$  est fini.
- $I_{<\omega}(S_{\alpha})$  où  $S_{\alpha}$  est une sierpinskisation de  $\alpha$  et de  $\omega$  si  $\alpha$  est infini dénombrable.

Ce résultat est conséquence des deux théorèmes suivants:

**Théorème 0.5.** (cf. Theorem 2.5, Chapter 2) Soit  $\alpha$  un type d'ordre dénombrable. Le sup-treillis  $[\omega]^{<\omega}$  est minimal dans  $\mathbb{J}_\alpha$  si et seulement si  $\alpha$  n'est pas un ordinal.

Soit  $\alpha$  un ordinal. Soit  $S_\alpha := \alpha$  si  $\alpha < \omega$ . Si  $\alpha = \omega\alpha' + n$  avec  $\alpha' \neq 0$  et  $n < \omega$ , soit  $S_\alpha := \Omega(\alpha') \oplus n$  la somme directe de  $\Omega(\alpha')$  et de la chaîne  $n$ , où  $\Omega(\alpha')$  est une sierpinskisation de  $\omega\alpha'$  et  $\omega$ , via une bijection  $\varphi : \omega\alpha' \rightarrow \omega$  telle que  $\varphi^{-1}$  soit croissante sur chaque  $\omega \cdot \{\beta\}$ .

**Théorème 0.6.** (cf. Theorem 2.6, Chapter 2). Si  $\alpha$  est un ordinal, alors  $Q_\alpha := I_{<\omega}(S_\alpha)$  est le plus petit sous sup-treillis  $P$  de  $[\omega]^{<\omega}$  qui appartient à  $\mathbb{J}_\alpha$ .

Les ensembles ordonnés  $\underline{\Omega}(\omega^*)$  et  $\underline{\Omega}(\eta)$  sont des sup-treillis. Pour chaque chaîne dénombrable  $\alpha$ , nous considérons des instances particulières de  $\Omega(\alpha)$  qui sont des sous sup-treillis de  $\omega \times \alpha$ . Nous désignons encore par  $\Omega(\alpha)$  l'un quelconque d'entre eux, comme nous désignons par  $\underline{\Omega}(\alpha)$  le sup-treillis obtenu en ajoutant un plus petit élément à  $\Omega(\alpha)$  s'il n'en a pas déjà un.

A un type d'ordre dénombrable  $\alpha$  nous associons un sup-treillis  $P_\alpha$  défini comme suit.

Ou bien  $\alpha$  ne s'abrite pas dans une de ses sections finales strictes. Dans ce cas  $\alpha$  s'écrit  $\alpha = n + \alpha'$  avec  $n < \omega$  et  $\alpha'$  sans premier élément. Et nous posons  $P_\alpha := n + \underline{\Omega}(\alpha')$ . Dans le cas contraire, si  $\alpha$  est équimorphe à  $\omega + \alpha'$  nous posons  $P_\alpha := \underline{\Omega}(1 + \alpha')$ , sinon, nous posons  $P_\alpha = \underline{\Omega}(\alpha)$ .

L'importance du type de sierpinskisation ci-dessus vient du résultat suivant:

**Théorème 0.7.** (cf. Theorem 2.9, Chapter 2) Si  $\alpha$  est un type d'ordre infini dénombrable,  $P_\alpha$  est minimal dans  $\mathbb{J}_\alpha$ .

Des théorèmes 0.5 et 0.7 découle que:

Si  $\alpha$  n'est pas un ordinal,  $P_\alpha$  et  $[\omega]^{<\omega}$  sont deux éléments incomparables de  $\mathbb{J}_\alpha$ .

Tandis que d'après les théorèmes 0.6 et 0.7:

Si  $\alpha$  est un ordinal infini dénombrable,  $P_\alpha$  et  $Q_\alpha$  sont des obstructions minimales.

En fait, si  $\alpha \leq \omega + \omega = \omega 2$ , elles coïncident. En effet, si  $\alpha = \omega + n$  avec  $n < \omega$ , alors  $S_\alpha = \Omega(1) \oplus n$ . Comme  $\Omega(1)$  est isomorphe à  $\omega$ ,  $Q_\alpha$  est isomorphe au produit direct  $\omega \times (n+1)$  qui est lui même isomorphe à  $\Omega(n+1) = P_\alpha$ . Si  $\alpha = \omega + \omega = \omega 2$ ,  $S_\alpha = \Omega(2)$ . Cet ensemble ordonné est isomorphe au produit direct  $\omega \times 2$ . Et l'ensemble  $Q_\alpha$  est isomorphe à  $[\omega]^2$ , la partie du produit direct  $\omega \times \omega$  constituée des couples  $(i, j)$  tels que  $i < j$ . En retour  $[\omega]^2$  est isomorphe à  $\Omega(\omega) = P_\alpha$ .

Comme nous le verrons (Chapitre 2, Corollary 2.1) au delà de  $\omega 2$  ces deux obstructions sont incomparables.

Ce travail suggère deux autres questions.

**QUESTIONS 0.4.** (1) Si  $\alpha$  est un ordinal infini, est ce que les obstructions minimales sont  $\alpha$ ,  $P_\alpha$ ,  $Q_\alpha$  et des sommes lexicographiques d'obstructions correspondant à des ordinaux plus petits?

(2) Si  $\alpha$  est une chaîne dispersée non bien ordonnée, est ce que les obstructions minimales sont  $\alpha$ ,  $P_\alpha$ ,  $[\omega]^{<\omega}$  et des sommes lexicographiques d'obstructions correspondant à chaînes dispersées plus petites?



Nous ne pouvons donner que quelques exemples de type d'ordres pour lesquels la réponse est positive.

## 2. Le cas des treillis modulaires

N'ayant pas de réponse au problème posé, nous le restreignons. Au lieu de considérer la classe des treillis algébriques nous considérons des sous-classes particulières. Une classe intéressante est celle des treillis algébriques modulaires. Au lieu de rechercher les obstructions correspondant à un type dénombrable  $\alpha$  nous recherchons les types d'ordre  $\alpha$  pour lesquels il n'y a que deux obstructions, à savoir  $1 + \alpha$  et  $[\omega]^{<\omega}$ . Les résultats que nous obtenons sont valables pour une classe de treillis plus large que les treillis modulaires algébriques, la classe  $\mathbb{L}$  définie comme suit. Notons  $\mathbb{A}$  la classe des treillis algébriques. Pour un type d'ordre  $\alpha$ , soit  $L_\alpha := 1 + (1 \oplus \alpha) + 1$  le treillis formé par la somme directe de la chaîne à un élément 1 et de la chaîne  $\alpha$ , avec un plus petit et un plus grand éléments ajoutés. Notons  $\mathbb{L}$  la collection des  $L \in \mathbb{A}$  tel que  $L$  ne contient aucun sous-treillis isomorphe à  $L_{\omega+1}$  ou à  $L_{\omega^*}$ . Puisque  $L_2$  est isomorphe à  $M_5$ , le treillis non modulaire à cinq éléments, tout treillis algébrique modulaire appartient à  $\mathbb{L}$ .

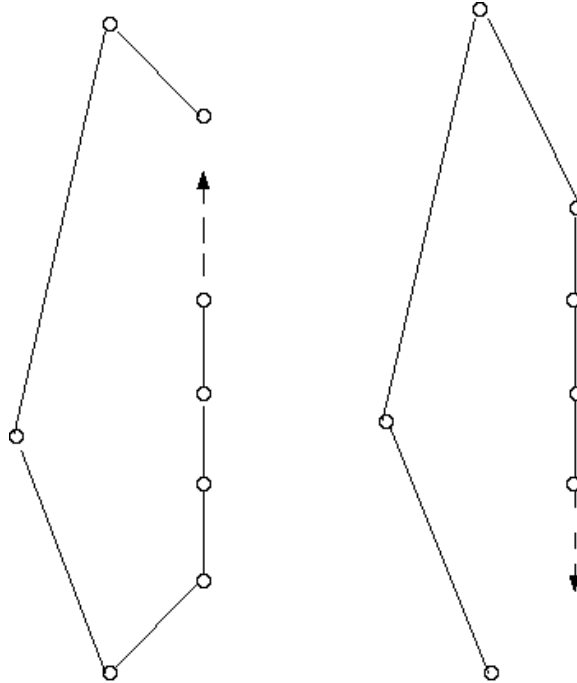
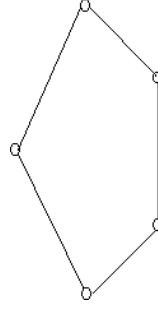


FIGURE 0.3.  $L_{\omega+1}$ ,  $L_{\omega^*}$

Pour un type d'ordre  $\alpha$ , nous notons  $\mathbb{A}_{-\alpha}$  (resp.  $\mathbb{L}_{-\alpha}$ ) la collection des  $L \in \mathbb{A}$  (resp.  $L \in \mathbb{L}$ ) tel que  $I(\alpha) \not\leq L$ . Considérons la classe  $\mathbb{K}$  des types d'ordre  $\alpha$  tel que  $L \in \mathbb{L}_{-\alpha}$  dès que  $K(L)$  ne contient ni chaîne de type  $1 + \alpha$  ni sous ensemble isomorphe à  $[\omega]^{<\omega}$ . Si  $\alpha$  est dénombrable alors ces deux conditions sont nécessaires

FIGURE 0.4.  $M_5$ 

pour interdire  $I(\alpha)$  dans  $L$ ; si, de plus,  $\alpha$  est indécomposable, cela revient à interdire  $\alpha$  dans  $L$ .

**Théorème 0.8.** (cf. Theorem 3.1, Chapter 3) La classe  $\mathbb{K}$  satisfait les propriétés suivantes:

- ( $p_1$ )  $0 \in \mathbb{K}$ ,  $1 \in \mathbb{K}$ ;
- ( $p_2$ ) Si  $\alpha + 1 \in \mathbb{K}$  et  $\beta \in \mathbb{K}$  alors  $\alpha + 1 + \beta \in \mathbb{K}$ ;
- ( $p_3$ ) Si  $\alpha_n + 1 \in \mathbb{K}$  pour tout  $n < \omega$  alors la  $\omega$ -somme  $\gamma := \alpha_0 + 1 + \alpha_1 + 1 + \dots + \alpha_n + 1 + \dots$  est dans  $\mathbb{K}$ ;
- ( $p_4$ ) Si  $\alpha_n \in \mathbb{K}$  et  $\{m : \alpha_n \leq \alpha_m\}$  est infini pour tout  $n < \omega$  alors la  $\omega^*$ -somme  $\delta := \dots + \alpha_{n+1} + \alpha_n + \dots + \alpha_1 + \alpha_0$  est dans  $\mathbb{K}$ .
- ( $p_5$ ) Si  $\alpha$  est un type d'ordre dénombrable dispersé, alors  $\alpha \in \mathbb{K}$  si et seulement si  $\alpha = \alpha_0 + \alpha_1 + \dots + \alpha_n$  avec  $\alpha_i \in \mathbb{K}$  pour  $i \leq n$ ,  $\alpha_i$  strictement indécomposable à gauche pour  $i < n$  et  $\alpha_n$  indécomposable;
- ( $p_6$ )  $\eta \in \mathbb{K}$ .

**Corollaire 0.1.** (cf. Corollary 3.1, Chapter 3) Un treillis algébrique modulaire est bien fondé, respectivement dispersé, si et seulement si le sup-treillis de ses éléments compacts est bien fondé, resp. dispersé, et ne contient pas de sous sup-treillis isomorphe à  $[\omega]^{<\omega}$ .

On peut dire un peu plus. Soit  $\mathbb{P}$  la plus petite classe de types d'ordre, satisfaisant les propriétés ( $p_1$ ) à ( $p_4$ ) dessus. Par exemple,  $\omega, \omega^*, \omega(\omega^\alpha)^*, \omega^*\omega, \omega^*\omega\omega^*$  sont dans  $\mathbb{P}$ . On a  $\mathbb{P} \subseteq \mathbb{K} \setminus \{\eta\}$ .

Nous ignorons si tout type d'ordre dénombrable  $\alpha \in \mathbb{K} \setminus \{\eta\}$  est équimorphe à un  $\alpha' \in \mathbb{P}$ . La classe des types d'ordre  $\alpha$  qui sont équimorphes à un  $\alpha' \in \mathbb{P}$  satisfait aussi ( $p_5$ ) donc, pour répondre à notre question, nous pouvons supposer que  $\alpha$  est indécomposable. Nous n'avons réussi à répondre à cette question que dans le cas où  $\alpha$  est indivisible, c'est à dire s'abrite dans au moins une des deux parties de n'importe quelle partition de  $\alpha$  en deux parties. Nous éliminons les types d'ordres indivisibles qui ne sont pas dans  $\mathbb{P}$  en construisant des treillis algébriques de la forme  $J(T)$  où  $T$  est un treillis distributif approprié. Ces treillis distributifs sont obtenus par des sierpinskiations. Nous obtenons:

**Théorème 0.9.** (cf. Theorem 3.2, Chapter 3) *Un type d'ordre dénombrable indivisible  $\alpha$  est équimorphe à un type d'ordre dénombrable  $\alpha' \in \mathbb{P} \cup \{\eta\}$  si et seulement si quelque soit le treillis algébrique modulaire  $L$  les propriétés suivantes sont équivalentes:*

- (i)  *$L$  ne contient pas de chaîne de type  $I(\alpha)$ .*
- (ii) *Le sup-treillis  $K(L)$  des éléments compacts ne contient ni  $1 + \alpha$  ni de sous sup-lattice isomorphe à  $[\omega]^{<\omega}$ .*

### 3. Treillis distributifs et sup-treillis $[\omega]^{<\omega}$

Le fait qu'un sup-treillis  $P$  contienne un sous-sup-treillis isomorphe à  $[\omega]^{<\omega}$  équivaut à l'existence d'un ensemble indépendant infini. Rappelons que dans un sup-treillis  $P$ , un sous-ensemble  $X$  est *indépendant* si  $x \notin \bigvee F$  pour tout  $x \in X$  et toute partie finie non vide  $F$  de  $X \setminus \{x\}$ . Les conditions qui assurent l'existence d'un ensemble indépendant infini ou les conséquences de leurs inexistence ont été déjà considérés (voir les recherches sur le "free-subset problem" de Hajnal [39] ou "on the cofinality of posets" [14, 27]).

Le résultat suivant, traduit l'existence d'un ensemble indépendant de taille  $\kappa$  pour un cardinal  $\kappa$ .

**Théorème 0.10.** [8] [25] (cf. Theorem 4.2, Chapter 4) *Soit  $\kappa$  un cardinal ; pour un sup-treillis  $P$  les propriétés suivantes sont équivalentes:*

- (i)  *$P$  contient un ensemble indépendant de taille  $\kappa$ ;*
- (ii)  *$P$  contient un sous sup-treillis isomorphe à  $[\kappa]^{<\omega}$ ;*
- (iii)  *$P$  contient un sous-ensemble isomorphe à  $[\kappa]^{<\omega}$ ;*
- (iv)  *$J(P)$  contient un sous-ensemble isomorphe à  $\mathfrak{P}(\kappa)$ ;*
- (v)  *$\mathfrak{P}(\kappa)$  se plonge dans  $J(P)$  par une application qui préserve les sup arbitraires.*

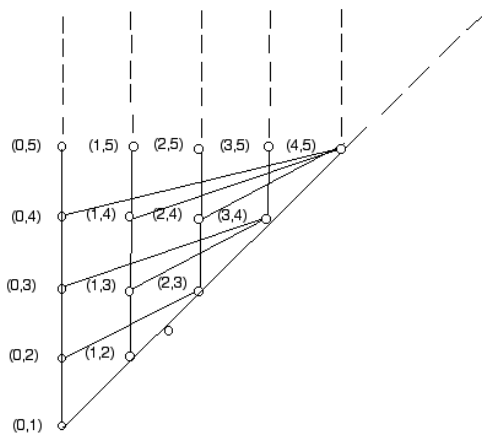
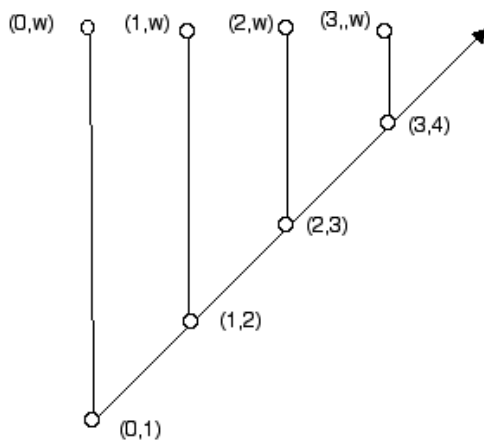
L'existence d'un ensemble indépendant infini, dans le cas d'un treillis distributif, se traduit par l'existence de deux sup-treillis particuliers  $I_{<\omega}(\Gamma)$  et  $I_{<\omega}(\Delta)$ , où  $\Gamma$  et  $\Delta$  sont deux inf-treillis spéciaux.

Soit  $\Delta := \{(i, j) : i < j \leq \omega\}$  muni de l'ordre  $(i, j) \leq (i', j')$  si et seulement si  $j \leq i'$  ou  $i = i'$  et  $j \leq j'$ . Soit  $\Gamma := \{(i, j) \in \Delta : j = i + 1 \text{ ou } j = \omega\}$  muni de l'ordre induit. Les ensembles ordonnés  $\Delta$  et  $\Gamma$  sont des inf-treillis bien-fondés dont les éléments maximaux forment une antichaîne infinie. Pour un ensemble ordonné  $Q$ , notons  $I_{<\omega}(Q)$  l'ensemble des sections initiales finiment engendrés de  $Q$ . Les ensembles  $I_{<\omega}(\Delta)$  et  $I_{<\omega}(\Gamma)$ , sont des treillis distributifs bien-fondés contenant un sous ensemble isomorphe à  $[\omega]^{<\omega}$ .

Notons que, pour un treillis distributif, contenir  $[\omega]^{<\omega}$ , comme sous sup-treillis ou comme sous-ensemble ordonné est équivalent.

**Théorème 0.11.** (cf. Theorem 4.3, Chapter 4) *Soit  $T$  un treillis distributif. Les propriétés suivantes sont équivalentes:*

- (i)  *$T$  contient un sous-ensemble isomorphe à  $[\omega]^{<\omega}$ ;*
- (ii) *Le treillis  $[\omega]^{<\omega}$  est quotient d'un sous-treillis de  $T$ ;*
- (iii)  *$T$  contient un sous-treillis isomorphe à  $I_{<\omega}(\Gamma)$  ou à  $I_{<\omega}(\Delta)$ .*

$(0,w) \circ (1,w) \circ (2,w) \circ (3,w) \circ (4,w) \circ$ 
FIGURE 0.5.  $\Delta$ FIGURE 0.6.  $\Gamma$ 

Un des arguments de la preuve du Théorème 0.11 est le Théorème de Ramsey [37] appliqué comme dans [11]. Les ensembles ordonnés  $\Delta$  (avec un plus grand élément ajouté) et  $\Gamma$  ont été considérés avant, dans [23] et [24]. L'ensemble ordonné  $\delta$  obtenu à partir de  $\Delta$  en supprimant les éléments maximaux a été aussi considéré par E. Corominas en 1970 comme une variante de l'exemple construit par R. Rado [36].

## CHAPTER 1

### A characterization of well-founded algebraic lattices

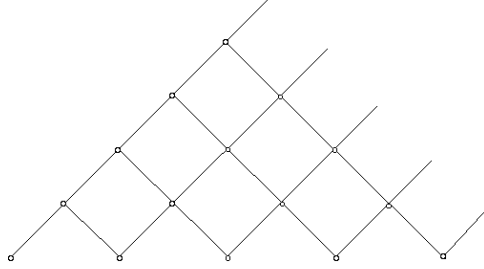
We characterize well-founded algebraic lattices by means of forbidden subsemilattices of the join-semilattice made of their compact elements. More specifically, we show that an algebraic lattice  $L$  is well-founded if and only if  $K(L)$ , the join-semilattice of compact elements of  $L$ , is well-founded and contains neither  $[\omega]^{<\omega}$ , nor  $\underline{\Omega}(\omega^*)$  as a join-subsemilattice. As an immediate corollary, we get that an algebraic modular lattice  $L$  is well-founded if and only if  $K(L)$  is well-founded and contains no infinite independent set. If  $K(L)$  is a join-subsemilattice of  $I_{<\omega}(Q)$ , the set of finitely generated initial segments of a well-founded poset  $Q$ , then  $L$  is well-founded if and only if  $K(L)$  is well-quasi-ordered.

#### 1. Introduction and synopsis of results

Algebraic lattices and join-semilattices (with a 0) are two aspects of the same thing, as expressed in the following basic result.

**Theorem 1.1.** [19], [15] *The collection  $J(P)$  of ideals of a join-semilattice  $P$ , once ordered by inclusion, is an algebraic lattice and the subposet  $K(J(P))$  of its compact elements is isomorphic to  $P$ . Conversely, the subposet  $K(L)$  of compact elements of an algebraic lattice  $L$  is a join-semilattice with a 0 and  $J(K(L))$  is isomorphic to  $L$ .*

In this paper, we characterize well-founded algebraic lattices by means of forbidden join-subsemilattices of the join-semilattice made of their compact elements. In the sequel  $\omega$  denotes the chain of non-negative integers, and when this causes no confusion, the first infinite cardinal as well as the first infinite ordinal. We denote  $\omega^*$  the chain of negative integers. We recall that a poset  $P$  is *well-founded* provided that every non-empty subset of  $P$  has a minimal element. With the Axiom of dependent choices, this amounts to the fact that  $P$  contains no subset isomorphic to  $\omega^*$ . Let  $\Omega(\omega^*)$  be the set  $[\omega]^2$  of two-element subsets of  $\omega$ , identified to pairs  $(i, j)$ ,  $i < j < \omega$ , ordered so that  $(i, j) \leq (i', j')$  if and only if  $i' \leq i$  and  $j \leq j'$  w.r.t. the natural order on  $\omega$ . Let  $\underline{\Omega}(\omega^*) := \Omega(\omega^*) \cup \{\emptyset\}$  be obtained by adding a least element. Note that  $\underline{\Omega}(\omega^*)$  is isomorphic to the set of bounded intervals of  $\omega$  (or  $\omega^*$ ) ordered by inclusion. Moreover  $\underline{\Omega}(\omega^*)$  is a join-semilattice  $((i, j) \vee (i', j') = (i \wedge i', j \vee j'))$ . The join-semilattice  $\underline{\Omega}(\omega^*)$  embeds in  $\Omega(\omega^*)$  as a join-semilattice; the advantage of  $\underline{\Omega}(\omega^*)$  w.r.t. our discussion is to have a zero. Let  $\kappa$  be a cardinal number, e.g.  $\kappa := \omega$ ; denote  $[\kappa]^{<\omega}$  (resp.  $\mathfrak{P}(\kappa)$ ) the set, ordered by inclusion, consisting of finite (resp. arbitrary) subsets of  $\kappa$ . The posets  $\underline{\Omega}(\omega^*)$  and  $[\kappa]^{<\omega}$  are well-founded lattices, whereas the algebraic lattices  $J(\underline{\Omega}(\omega^*))$  and  $J([\kappa]^{<\omega})$  ( $\kappa$  infinite) are not

FIGURE 1.1.  $\Omega(\omega^*)$ 

well-founded (and we may note that  $J([\kappa]^{<\omega})$  is isomorphic to  $\mathfrak{P}(\kappa)$ ). As a poset  $\underline{\Omega}(\omega^*)$  is isomorphic to a subset of  $[\omega]^{<\omega}$ , but not as a join-subsemilattice. This is our first result.

**Proposition 1.1.**  *$\underline{\Omega}(\omega^*)$  does not embed in  $[\omega]^{<\omega}$  as a join-subsemilattice; more generally, if  $Q$  is a well-founded poset then  $\underline{\Omega}(\omega^*)$  does not embed as a join-subsemilattice into  $I_{<\omega}(Q)$ , the join-semilattice made of finitely generated initial segments of  $Q$ .*

Our next result expresses that  $\underline{\Omega}(\omega^*)$  and  $[\omega]^{<\omega}$  are unavoidable examples of well-founded join-semilattices whose set of ideals is not well-founded.

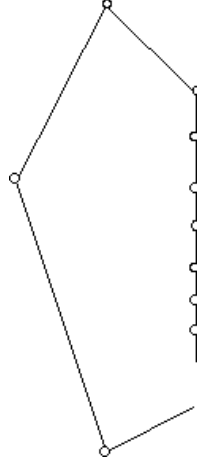
**Theorem 1.2.** *An algebraic lattice  $L$  is well-founded if and only if  $K(L)$  is well-founded and contains no join-subsemilattice isomorphic to  $\underline{\Omega}(\omega^*)$  or to  $[\omega]^{<\omega}$ .*

The fact that a join-semilattice  $P$  contains a join-subsemilattice isomorphic to  $[\omega]^{<\omega}$  amounts to the existence of an infinite independent set. Let us recall that a subset  $X$  of a join-semilattice  $P$  is *independent* if  $x \notin \bigvee F$  for every  $x \in X$  and every non-empty finite subset  $F$  of  $X \setminus \{x\}$ . Conditions which may insure the existence of an infinite independent set or consequences of the inexistence of such sets have been considered within the framework of the structure of closure systems (cf. the research on the "free-subset problem" of Hajnal [39] or on the cofinality of posets [14, 27]). A basic result is the following.

**Theorem 1.3.** [8] [25] *Let  $\kappa$  be a cardinal number; for a join-semilattice  $P$  the following properties are equivalent:*

- (i)  *$P$  contains an independent set of size  $\kappa$ ;*
- (ii)  *$P$  contains a join-subsemilattice isomorphic to  $[\kappa]^{<\omega}$ ;*
- (iii)  *$P$  contains a subposet isomorphic to  $[\kappa]^{<\omega}$ ;*
- (iv)  *$J(P)$  contains a subposet isomorphic to  $\mathfrak{P}(\kappa)$ ;*
- (v)  *$\mathfrak{P}(\kappa)$  embeds in  $J(P)$  via a map preserving arbitrary joins.*

Let  $L(\alpha) := 1 + (1 \oplus J(\alpha)) + 1$  be the lattice made of the direct sum of the one-element chain 1 and the chain  $J(\alpha)$ , ( $\alpha$  finite or equal to  $\omega^*$ ), with top and bottom added.

FIGURE 1.2.  $L(\omega^*)$ 

Clearly  $J(\underline{\Omega}(\omega^*))$  contains a sublattice isomorphic to  $L(\omega^*)$ . Since a modular lattice contains no sublattice isomorphic to  $L(2)$ , we get as a corollary of Theorem 1.2:

**Theorem 1.4.** *An algebraic modular lattice  $L$  is well-founded if and only if  $K(L)$  is well-founded and contains no infinite independent set.*

Another consequence is this:

**Theorem 1.5.** *For a join-semilattice  $P$ , the following properties are equivalent:*

- (i)  $P$  is well-founded with no infinite antichain ;
- (ii)  $P$  contains no infinite independent set and embeds as a join-semilattice into a join-semilattice of the form  $I_{<\omega}(Q)$  where  $Q$  is some well-founded poset.

Posets which are well-founded and have no infinite antichain are said *well-partially-ordered* or *well-quasi-ordered*, wqo for short. They play an important role in several areas (see [13]). If  $P$  is a wqo join-semilattice then  $J(P)$ , the poset of ideals of  $P$ , is well-founded and one may assign to every  $J \in J(P)$  an ordinal, its *height*, denoted by  $h(J, J(P))$ . This ordinal is defined by induction, setting  $h(J, J(P)) := \text{Sup}(\{h(J', J(P)) + 1 : J' \in J(P), J' \subset J\})$  and  $h(J', J(P)) := 0$  if  $J'$  is minimal in  $J(P)$ . The ordinal  $h(J(P)) := h(P, J(P)) + 1$  is the *height* of  $J(P)$ . If  $P := I_{<\omega}(Q)$ , with  $Q$  wqo, then  $J(P)$  contains a chain of order type  $h(J(P))$ . This is an equivalent form of the famous result of de Jongh and Parikh [20] asserting that among the linear extensions of a wqo, one has a maximum order type.

**PROBLEM 1.1.** *Let  $P$  be a wqo join-semilattice; does  $J(P)$  contain a chain of order type  $h(J(P))$ ?*

An immediate corollary of Theorem 1.5 is:

**Corollary 1.1.** *A join-semilattice  $P$  of  $[\omega]^{<\omega}$  contains either  $[\omega]^{<\omega}$  as a join-semilattice or is wqo.*

Let us compare join-subsemilattices of  $[\omega]^{<\omega}$ . Set  $P \leq P'$  for two such join-subsemilattices if  $P$  embeds in  $P'$  as a join-semilattice. This gives a quasi-order and, according to Corollary 1.1, the poset corresponding to this quasi-order has a largest element (namely  $[\omega]^{<\omega}$ ), and all other members come from wqo join-semilattices. Basic examples of join-subsemilattices of  $[\omega]^{<\omega}$  are the  $I_{<\omega}(Q)$ 's where  $Q$  is a countable poset such that no element is above infinitely many elements. These posets  $Q$  are exactly those which are embeddable in the poset  $[\omega]^{<\omega}$  ordered by inclusion. An interesting subclass is made of posets of the form  $Q = (\mathbb{N}, \leq)$  where the order  $\leq$  is the intersection of the natural order  $\mathfrak{N}$  on  $\mathbb{N}$  and of a linear order  $\mathfrak{L}$  on  $\mathbb{N}$ , (that is  $x \leq y$  if  $x \leq y$  w.r.t.  $\mathfrak{N}$  and  $x \leq y$  w.r.t.  $\mathfrak{L}$ ). If  $\alpha$  is the type of the linear order, a poset of this form is a *sierpinskisation* of  $\alpha$ . The corresponding join-semilattices are wqo provided that the posets  $Q$  have no infinite antichain; in the particular case of a sierpinskisation of  $\alpha$  this amounts to the fact that  $\alpha$  is well-ordered.

As shown in [34], sierpinskisations given by a bijective map  $\psi : \omega\alpha \rightarrow \omega$  which is order-preserving on each component  $\omega \cdot \{i\}$  of  $\omega\alpha$  are all embeddable in each other, and for this reason denoted by the same symbol  $\Omega(\alpha)$ . Among the representatives of  $\Omega(\alpha)$ , some are join-semilattices, and among them, join-subsemilattices of the direct product  $\omega \times \alpha$  (this is notably the case of the poset  $\Omega(\omega^*)$  we previously defined). We extend the first part of Proposition 2.1, showing that except for  $\alpha \leq \omega$ , the representatives of  $\Omega(\alpha)$  which are join-semilattices never embed in  $[\omega]^{<\omega}$  as join-semilattices, whereas they embed as posets (see Corollary 1.4 and Example 1.2). From this result, it follows that the posets  $\Omega(\alpha)$  and  $I_{<\omega}(\Omega(\alpha))$  do not embed in each other as join-semilattices.

These two posets provide examples of a join-semilattice  $P$  such that  $P$  contains no chain of type  $\alpha$  while  $J(P)$  contains a chain of type  $J(\alpha)$ . However, if  $\alpha$  is not well ordered then  $I_{<\omega}(\Omega(\alpha))$  and  $[\omega]^{<\omega}$  embed in each other as join-semilattices.

**PROBLEM 1.2.** *Let  $\alpha$  be a countable ordinal. Is there a minimum member among the join-subsemilattices  $P$  of  $[\omega]^{<\omega}$  such that  $J(P)$  contains a chain of type  $\alpha + 1$ ? Is it true that this minimum is  $I_{<\omega}(\Omega(\alpha))$  if  $\alpha$  is indecomposable?*

## 2. Definitions and basic results

Our definitions and notations are standard and agree with [15] except on minor points that we will mention. We adopt the same terminology as in [8]. We recall only few things. Let  $P$  be a poset. A subset  $I$  of  $P$  is an *initial segment* of  $P$  if  $x \in P$ ,  $y \in I$  and  $x \leq y$  imply  $x \in I$ . If  $A$  is a subset of  $P$ , then  $\downarrow A = \{x \in P : x \leq y \text{ for some } y \in A\}$  denotes the least initial segment containing  $A$ . If  $I = \downarrow A$  we say that  $I$  is *generated* by  $A$  or  $A$  is *cofinal* in  $I$ . If  $A = \{a\}$  then  $I$  is a *principal initial segment* and we write  $\downarrow a$  instead of  $\downarrow \{a\}$ . We denote  $\text{down}(P)$  the set of principal initial segments of  $P$ . A *final segment* of  $P$  is any initial segment of  $P^*$ , the dual of  $P$ . We denote by  $\uparrow A$  the final segment generated by  $A$ . If  $A = \{a\}$  we write  $\uparrow a$  instead of  $\uparrow \{a\}$ . A subset  $I$  of  $P$  is *directed* if every finite subset of  $I$  has an upper bound in  $I$  (that is  $I$  is non-empty and every pair of elements of  $I$  has an upper bound). An *ideal* is a non-empty directed initial segment of  $P$  (in some other



texts, the empty set is an ideal). We denote  $I(P)$  (respectively,  $I_{<\omega}(P)$ ,  $J(P)$ ) the set of initial segments (respectively, finitely generated initial segments, ideals of  $P$ ) ordered by inclusion and we set  $J_*(P) := J(P) \cup \{\emptyset\}$ ,  $I_0(P) := I_{<\omega}(P) \setminus \{\emptyset\}$ . Others authors use *down set* for initial segment. Note that  $\text{down}(P)$  has not to be confused with  $I(P)$ . If  $P$  is a join-semilattice with a 0, an element  $x \in P$  is *join-irreducible* if it is distinct from 0, and if  $x = a \vee b$  implies  $x = a$  or  $x = b$  (this is a slight variation from [15]). We denote  $\mathbb{J}_{\text{irr}}(P)$  the set of join-irreducibles of  $P$ . An element  $a$  in a lattice  $L$  is *compact* if for every  $A \subset L$ ,  $a \leq \bigvee A$  implies  $a \leq \bigvee A'$  for some finite subset  $A'$  of  $A$ . The lattice  $L$  is *compactly generated* if every element is a supremum of compact elements. A lattice is *algebraic* if it is complete and compactly generated.

We note that  $I_{<\omega}(P)$  is the set of compact elements of  $I(P)$ , hence  $J(I_{<\omega}(P)) \cong I(P)$ . Moreover  $I_{<\omega}(P)$  is a lattice, and in fact a distributive lattice, if and only if  $P$  is  $\downarrow$ -closed, that is, the intersection of two principal initial segments of  $P$  is a finite union, possibly empty, of principal initial segments. We also note that  $J(P)$  is the set of join-irreducible elements of  $I(P)$ ; moreover,  $I_{<\omega}(J(P)) \cong I(P)$  whenever  $P$  has no infinite antichain.

Notably for the proof of Theorem 1.7, we will need the following results.

**Theorem 1.6.** *Let  $P$  be a poset.*

- a)  $I_{<\omega}(P)$  is well-founded if and only if  $P$  is well-founded (Birkhoff 1937, see [1]);
- b)  $I_{<\omega}(P)$  is wqo iff  $P$  is wqo iff  $I(P)$  well-founded (Higman 1952 [17]);
- c) if  $P$  is a well-founded join-semilattice with a 0, then every member of  $P$  is a finite join of join-irreducible elements of  $P$  (Birkhoff, 1937, see [1]);
- d) A join-semilattice  $P$  with a zero is wqo if and only if every member of  $P$  is a finite join of join-irreducible elements of  $P$  and the set  $\mathbb{J}_{\text{irr}}(P)$  of these join-irreducible elements is wqo (follows from b) and c)).

A poset  $P$  is *scattered* if it does not contain a copy of  $\eta$ , the chain of rational numbers. A topological space  $T$  is *scattered* if every non-empty closed set contains some isolated point. The power set of a set, once equipped with the product topology, is a compact space. The set  $J(P)$  of ideals of a join-semilattice  $P$  with a 0 is a closed subspace of  $\mathfrak{P}(P)$ , hence is a compact space too. Consequently, an algebraic lattice  $L$  can be viewed as a poset and a topological space as well. It is easy to see that if  $L$  is topologically scattered then it is order scattered. It is a more significant fact, due to M.Mislove [28], that the converse holds if  $L$  is distributive.

### 3. Separating chains of ideals and proofs of Proposition 2.1 and Theorem 1.2

Let  $P$  be a join-semilattice. If  $x \in P$  and  $J \in J(P)$ , then  $\downarrow x$  and  $J$  have a join  $\downarrow x \vee J$  in  $J(P)$  and  $\downarrow x \vee J = \downarrow \{x \vee y : y \in J\}$ . Instead of  $\downarrow x \vee J$  we also use the notation  $\{x\} \vee J$ . Note that  $\{x\} \vee J$  is the least member of  $J(P)$  containing  $\{x\} \cup J$ . We say that a non-empty chain  $\mathcal{I}$  of ideals of  $P$  is *separating* if for every  $I \in \mathcal{I} \setminus \{\bigcup \mathcal{I}\}$  and every  $x \in \bigcup \mathcal{I} \setminus I$ , there is some  $J \in \mathcal{I}$  such that  $I \not\subseteq \{x\} \vee J$ . If  $\mathcal{I}$  is separating then  $\mathcal{I}$  has a least element implies it is a singleton set. In  $P := [\omega]^{<\omega}$ , the chain  $\mathcal{I} := \{I_n : n < \omega\}$  where  $I_n$  consists of the finite subsets of  $\{m :$

$n \leq m\}$  is separating. In  $P := \omega^*$ , the chain  $\mathcal{I} := \{\downarrow x : x \in P\}$  is non-separating, as well as all of its infinite subchains. In  $P := \Omega(\omega^*)$  the chain  $\mathcal{I} := \{I_n : n < \omega\}$  where  $I_n := \{(i, j) : n \leq i < j < \omega\}$  has the same property.

We may observe that a join-preserving embedding from a join-semilattice  $P$  into a join-semilattice  $Q$  transforms every separating (resp. non-separating) chain of ideals of  $P$  into a separating (resp. non-separating) chain of ideals of  $Q$  (If  $\mathcal{I}$  is a separating chain of ideals of  $P$ , then  $\mathcal{J} = \{f(I) : I \in \mathcal{I}\}$  is a separating chain of ideals of  $Q$ ). Hence the containment of  $[\omega]^{<\omega}$  (resp. of  $\omega^*$  or of  $\Omega(\omega^*)$ ), as a join-subsemilattice, provides a chain of ideals which is separating (resp. non-separating, as are all its infinite subchains, as well). We show in the next two lemmas that the converse holds.

**Lemma 1.1.** *A join-semilattice  $P$  contains an infinite independent set if and only if it contains an infinite separating chain of ideals.*

**Proof.** Let  $X = \{x_n : n < \omega\}$  be an infinite independent set. Let  $I_n$  be the ideal generated by  $X \setminus \{x_i : 0 \leq i \leq n\}$ . The chain  $\mathcal{I} = \{I_n : n < \omega\}$  is separating. Let  $\mathcal{I}$  be an infinite separating chain of ideals. Define inductively an infinite sequence  $x_0, I_0, \dots, x_n, I_n, \dots$  such that  $I_0 \in \mathcal{I} \setminus \{\cup \mathcal{I}\}$ ,  $x_0 \in \cup \mathcal{I} \setminus I_0$  and such that:

$a_n$ )  $I_n \in \mathcal{I}$ ;

$b_n$ )  $I_n \subset I_{n-1}$ ;

$c_n$ )  $x_n \in I_{n-1} \setminus (\{x_0 \vee \dots \vee x_{n-1}\} \vee I_n)$  for every  $n \geq 1$ .

The construction is immediate. Indeed, since  $\mathcal{I}$  is infinite then  $\mathcal{I} \setminus \{\cup \mathcal{I}\} \neq \emptyset$ . Choose arbitrary  $I_0 \in \mathcal{I} \setminus \{\cup \mathcal{I}\}$  and  $x_0 \in \cup \mathcal{I} \setminus I_0$ . Let  $n \geq 1$ . Suppose  $x_k, I_k$  defined and satisfying  $a_k), b_k), c_k)$  for all  $k \leq n-1$ . Set  $I := I_{n-1}$  and  $x := x_0 \vee \dots \vee x_{n-1}$ . Since  $I \in \mathcal{I}$  and  $x \in \cup \mathcal{I} \setminus I$ , there is some  $J \in \mathcal{I}$  such that  $I \not\subseteq \{x\} \vee J$ . Let  $z \in I \setminus (\{x\} \vee J)$ . Set  $x_n := z$ ,  $I_n := J$ . The set  $X := \{x_n : n < \omega\}$  is independent. Indeed if  $x \in X$  then since  $x = x_n$  for some  $n$ ,  $n < \omega$ , condition  $c_n$ ) asserts that there is some ideal containing  $X \setminus \{x\}$  and excluding  $x$ .  $\square$

**Lemma 1.2.** *A join-semilattice  $P$  contains either  $\omega^*$  or  $\Omega(\omega^*)$  as a join-subsemilattice if and only if it contains an  $\omega^*$ -chain  $\mathcal{I}$  of ideals such that all infinite subchains are non-separating.*

**Proof.** Let  $\mathcal{I}$  be an  $\omega^*$ -chain of ideals and let  $A$  be its largest element (that is  $A = \cup \mathcal{I}$ ). Let  $E$  denote the set  $\{x : x \in A \text{ and } I \subset \downarrow x \text{ for some } I \in \mathcal{I}\}$ .

**Case (i).** For every  $I \in \mathcal{I}$ ,  $I \cap E \neq \emptyset$ . We can build an infinite strictly decreasing sequence  $x_0, \dots, x_n, \dots$  of elements of  $P$ . Indeed, let us choose  $x_0 \in E \cap (\cup \mathcal{I})$  and  $I_0$  such that  $I_0 \subset \downarrow x_0$ . Suppose  $x_0, \dots, x_n$  and  $I_0, \dots, I_n$  defined such that  $I_i \subset \downarrow x_i$  for all  $i = 0, \dots, n$ . As  $E \cap I_n \neq \emptyset$  we can select  $x_{n+1} \in E \cap I_n$  and by definition of  $E$ , we can select some  $I_{n+1} \in \mathcal{I}$  such that  $I_{n+1} \subset \downarrow x_{n+1}$ . Thus  $\omega^* \leq P$ .

**Case (ii).** There is some  $I \in \mathcal{I}$  such that  $I \cap E = \emptyset$ . In particular all members of  $\mathcal{I}$  included in  $I$  are unbounded in  $I$ . Since all infinite subchains of  $\mathcal{I}$  are non-separating then, with no loss of generality, we may suppose that  $I = A$  (hence  $E = \emptyset$ ). We set  $I_{-1} := A$  and define a sequence  $x_0, I_0, \dots, x_n, I_n, \dots$  such that  $I_n \in \mathcal{I}$ ,  $x_n \in I_{n-1} \setminus I_n$  and  $I_n \subseteq \{x_n\} \vee I$  for all  $I \in \mathcal{I}$ , all  $n < \omega$ . Members of this

sequence being defined for all  $n', n' < n$ , observe that the set  $\mathcal{I}_n := \{I \in \mathcal{I} : I \subseteq I_{n-1}\}$  being infinite is non-separating, hence there are  $I \in \mathcal{I}_n$  and  $x \in I_{n-1} \setminus I$  such that  $I \subseteq \{x\} \vee J$  for all  $J \in \mathcal{I}_n$ . Set  $I_n := I$  and  $x_n := x$ . Next, we define a sequence  $y_0 := x_0, \dots, y_n, \dots$  such that for every  $n \geq 1$ :

$a_n$ )  $x_n \leq y_n \in I_{n-1}$ ;

$b_n$ )  $y_n \not\leq y_0 \vee y_{n-1}$ ;

$c_n$ )  $y_j \leq y_i \vee y_n$  for every  $i \leq j \leq n$ .

Suppose  $y_0, \dots, y_{n-1}$  defined for some  $n, n \geq 1$ . Since  $I_{n-1}$  is unbounded, we may select  $z \in I_{n-1}$  such that  $z \not\leq y_0 \vee \dots \vee y_{n-1}$ . If  $n = 1$ , we set  $y_1 := x_1 \vee z$ . Suppose  $n \geq 2$ . Let  $0 \leq j \leq n-2$ . Since  $y_{j+1} \vee \dots \vee y_{n-1} \in I_j \subseteq \{x_j\} \vee I_{n-1}$  we may select  $t_j \in I_{n-1}$  such that  $y_{j+1} \vee \dots \vee y_{n-1} \leq x_j \vee t_j$ . Set  $t := t_0 \vee \dots \vee t_{n-2}$  and  $y_n := x_n \vee z \vee t$ . Let  $f : \Omega(\omega^*) \rightarrow P$  be defined by  $f(i, j) := y_i \vee y_j$  for all  $(i, j), i < j < \omega$ .

Condition  $c_n$ ) insures that  $f$  is join-preserving. Indeed, let  $(i, j), (i', j') \in \Omega(\omega^*)$ . We have  $(i, j) \vee (i', j') = (i \wedge i', j \vee j')$  hence  $f((i, j) \vee (i', j')) = f(i \wedge i', j \vee j') = y_{i \wedge i'} \vee y_{j \vee j'}$ . If  $F$  is a finite subset of  $\omega$  with minimum  $a$  and maximum  $b$  then conditions  $c_n$ ) force  $\bigvee \{y_n : n \in F\} = y_a \vee y_b$ . If  $F := \{i, j, i', j'\}$  then, taking account of  $i < j$  and  $i' < j'$ , we have  $f(i, j) \vee f(i', j') = y_i \vee y_j \vee y_{i'} \vee y_{j'} = y_{i \wedge i'} \vee y_{j \vee j'}$ . Hence  $f((i, j) \vee (i', j')) = f(i, j) \vee f(i', j')$ , proving our claim.

Next,  $f$  is one-to-one. Let  $(i, j), (i', j') \in \Omega(\omega^*)$  such that  $f(i, j) = f(i', j')$ , that is  $y_i \vee y_j = y_{i'} \vee y_{j'}$  (1). Suppose  $j < j'$ . Since  $0 \leq i < j$ , Condition  $c_j$ ) implies  $y_i \leq y_0 \vee y_j$ . In the other hand, since  $0 \leq j \leq j' - 1$ , Condition  $c_{j'-1}$ ) implies  $y_j \leq y_0 \vee y_{j'-1}$ . Hence  $y_i \vee y_j \leq y_0 \vee y_{j'-1}$ . From (1) we get  $y_{j'} \leq y_0 \vee y_{j'-1}$ , contradicting Condition  $b_{j'}$ ). Hence  $j' \leq j$ . Exchanging the roles of  $j, j'$  gives  $j' \leq j$  thus  $j = j'$ . If  $i < i'$  then, Conditions  $a_{i'}$ ) and  $a_{j'}$ ) assure  $y_{i'} \in I_{i'-1}$  and  $y_{j'} \in I_{j'-1}$ . Since  $I_{j'-1} \subseteq I_{i'-1}$  we have  $y_{i'} \vee y_{j'} \in I_{i'-1}$ . In the other hand  $x_i \notin I_i$  and  $x_i \leq y_i \vee y_j$  thus  $y_i \vee y_j \notin I_i$ . From  $I_{i'-1} \subseteq I_i$ , we have  $y_i \vee y_j \notin I_{i'-1}$ , hence  $y_i \vee y_j \neq y_{i'} \vee y_{j'}$  and  $i' \leq i$ . Similarly we get  $i \leq i'$ . Consequently  $i = i'$ .  $\square$

**3.1. Proof of Proposition 2.1.** If  $\underline{\Omega}(\omega^*)$  embeds in  $[\omega]^{<\omega}$  then  $[\omega]^{<\omega}$  contains a non-separating  $\omega^*$ -chain of ideals. This is impossible: a non-separating chain of ideals of  $[\omega]^{<\omega}$  has necessarily a least element. Indeed, if the pair  $x, I$  ( $x \in [\omega]^{<\omega}$ ,  $I \in \mathcal{I}$ ) witnesses the fact that the chain  $\mathcal{I}$  is non-separating then there are at most  $|x| + 1$  ideals belonging to  $\mathcal{I}$  which are included in  $I$  (note that the set  $\{\cup I \setminus \cup J : J \subseteq I, J \in \mathcal{I}\}$  is a chain of subsets of  $x$ ). The proof of the general case requires more care. If  $\underline{\Omega}(\omega^*)$  embeds in  $I_{<\omega}(Q)$  as a join-semilattice then we may find a sequence  $x_0, I_0, \dots, x_n, I_n, \dots$  such that  $I_n \subset I_{n-1} \in J(I_{<\omega}(Q))$ ,  $x_n \in I_{n-1} \setminus I_n$  and  $I_n \subseteq \{x_n\} \vee I_m$  for every  $n < \omega$  and every  $m < \omega$ . Set  $I_\omega := \bigcap \{I_n : n < \omega\}$ ,  $\bar{I}_n := \cup I_n$  for every  $n \leq \omega$ ,  $Q' := Q \setminus \bar{I}_\omega$  and  $y_n := x_n \setminus \bar{I}_\omega$  for every  $n < \omega$ . We claim that  $y_0, \dots, y_n, \dots$  form a strictly descending sequence in  $I_{<\omega}(Q')$ . According to Property a) stated in Theorem 1.6,  $Q'$ , thus  $Q$ , is not well-founded.

First,  $y_n \in I_{<\omega}(Q')$ . Indeed, if  $a_n \in [Q]^{<\omega}$  generates  $x_n \in I_{<\omega}(Q)$  then, since  $\bar{I}_\omega \in I(Q)$ ,  $a_n \setminus \bar{I}_\omega$  generates  $x_n \setminus \bar{I}_\omega \in I(Q')$ . Next,  $y_{n+1} \subset y_n$ . It suffices to prove that the following inclusions hold:

$$x_{n+1} \cup \bar{I}_\omega \subseteq \bar{I}_n \subset x_n \cup \bar{I}_\omega$$

Indeed, subtracting  $\bar{I}_\omega$ , from the sets figuring above, we get:

$$y_{n+1} = (x_{n+1} \cup \bar{I}_\omega) \setminus \bar{I}_\omega \subset (x_n \cup \bar{I}_\omega) \setminus \bar{I}_\omega = y_n$$

The first inclusion is obvious. For the second note that, since  $J(I_{<\omega}(Q))$  is isomorphic to  $I(Q)$ , complete distributivity holds, hence with the hypotheses on the sequence  $x_0, I_0, \dots, x_n, I_n, \dots$  we have  $I_n \subseteq \bigcap \{\{x_n\} \vee I_m : m < \omega\} = \{x_n\} \vee \bigcap \{I_m : m < \omega\} = \{x_n\} \vee I_\omega$ , thus  $\bar{I}_n \subset x_n \cup \bar{I}_\omega$ .  $\square$

**Remark 1.1.** *One can deduce the fact that  $\Omega(\omega^*)$  does not embed as a join-semilattice in  $[\omega]^{<\omega}$  from the fact that it contains a strictly descending chain of completely meet-irreducible ideals (namely the chain  $\mathcal{I} := \{I_n : n < \omega\}$  where  $I_n := \{(i, j) : n \leq i < j < \omega\}$ ) (see Proposition 1.3) but this fact by itself does not prevent the existence of some well-founded poset  $Q$  such that  $\Omega(\omega^*)$  embeds as a join semilattice in  $I_{<\omega}(Q)$ .*

**3.2. Proof of Theorem 1.2.** In terms of join-semilattices and ideals, result becomes this: let  $P$  be a join-semilattice, then  $J(P)$  is well-founded if and only if  $P$  is well-founded and contains no join-subsemilattice isomorphic to  $\Omega(\omega^*)$  or to  $[\omega]^{<\omega}$ .

The proof goes as follows. Suppose that  $J(P)$  is not well-founded. If some  $\omega^*$ -chain in  $J(P)$  is separating then, according to Lemma 3.8,  $P$  contains an infinite independent set. From Theorem 1.3, it contains a join-subsemilattice isomorphic to  $[\omega]^{<\omega}$ . If no  $\omega^*$ -chain in  $J(P)$  is separating, then all the infinite subchains of an arbitrary  $\omega^*$ -chain are non-separating. From Lemma 1.2, either  $\omega^*$  or  $\Omega(\omega^*)$  embed in  $P$  as a join-semilattice. The converse is obvious.  $\square$

#### 4. Join-subsemilattices of $I_{<\omega}(Q)$ and proof of Theorem 1.5

In this section, we consider join-semilattices which embed in join-semilattices of the form  $I_{<\omega}(Q)$ . These are easy to characterize internally (see Proposition 1.2). This is also the case if the posets  $Q$  are antichains (see Proposition 1.3) but does not go so well if the posets  $Q$  are well-founded (see Lemma 1.1).

Let us recall that if  $P$  is a join-semilattice, an element  $x \in P$  is *join-prime* (or prime if there is no confusion), if it is distinct from the least element 0, if any, and if  $x \leq a \vee b$  implies  $x \leq a$  or  $x \leq b$ . This amounts to the fact that  $P \setminus \uparrow x$  is an ideal. We denote  $\mathbb{J}_{pri}(P)$ , the set of join-prime members of  $P$ . We recall that  $\mathbb{J}_{pri}(P) \subseteq \mathbb{J}_{irr}(P)$ ; the equality holds provided that  $P$  is a distributive lattice. It also holds if  $P = I_{<\omega}(Q)$ . Indeed:

**Fact 1.1.** *For an arbitrary poset  $Q$ , we have:*

$$(7) \quad \mathbb{J}_{irr}(I_{<\omega}(Q)) = \mathbb{J}_{pri}(I_{<\omega}(Q)) = \text{down}(Q)$$

**Fact 1.2.** *For a poset  $P$ , the following properties are equivalent:*

- $P$  is isomorphic to  $I_{<\omega}(Q)$  for some poset  $Q$ ;

- $P$  is a join-semilattice with a least element in which every element is a finite join of primes.

**Proof.** Observe that the primes in  $I_{<\omega}(Q)$ , are the  $\downarrow x$ ,  $x \in Q$ . Let  $I \in I_{<\omega}(Q)$  and  $F \in [Q]^{<\omega}$  generating  $I$ , we have  $I = \cup\{\downarrow x : x \in F\}$ . Conversely, let  $P$  be a join-semilattice with a 0. If every element in  $P$  is a finite join of primes, then  $P \cong I_{<\omega}(Q)$  where  $Q := \mathbb{J}_{pri}(P)$ .  $\square$

Let  $L$  be a complete lattice. For  $x \in L$ , set  $x^+ := \bigwedge\{y \in L : x < y\}$ . We recall that  $x \in L$  is *completely meet-irreducible* if  $x = \bigwedge X$  implies  $x \in X$ , or -equivalently-  $x \neq x^+$ . We denote  $\Delta(L)$  the set of completely meet-irreducible members of  $L$ . We recall the following Lemma.

**Lemma 1.3.** *Let  $P$  be a join-semilattice,  $I \in J(P)$  and  $x \in P$ . Then  $x \in I^+ \setminus I$  if and only if  $I$  is a maximal ideal of  $P \setminus \uparrow x$ .*

**Proposition 1.2.** *Let  $P$  be a join-semilattice. The following properties are equivalent:*

- (i)  $P$  embeds in  $I_{<\omega}(Q)$ , as a join-semilattice, for some poset  $Q$ ;
- (ii)  $P$  embeds in  $I_{<\omega}(J(P))$  as a join-semilattice;
- (iii)  $P$  embeds in  $I_{<\omega}(\Delta(J(P)))$  as a join-semilattice;
- (iv) For every  $x \in P$ ,  $P \setminus \uparrow x$  is a finite union of ideals.

**Proof.** (i)  $\Rightarrow$  (iv) Let  $\varphi$  be an embedding from  $P$  in  $P' := I_{<\omega}(Q)$ . We may suppose that  $P$  has a least element 0 and that  $\varphi(0) = \emptyset$  (if  $P$  has no least element, add one, say 0, and set  $\varphi(0) := \emptyset$ ; if  $P$  has a least element, say  $a$ , and  $\varphi(a) \neq \emptyset$ , add to  $P$  an element 0 below  $a$  and set  $\varphi(0) := \emptyset$ ). For  $J' \in \mathfrak{P}(P')$ , let  $\varphi^{-1}(J') := \{x \in P : \varphi(x) \in J'\}$ . Since  $\varphi$  is order-preserving,  $\varphi^{-1}(J') \in I(P)$  whenever  $J' \in I(P')$ ; moreover, since  $\varphi$  is join-preserving,  $\varphi^{-1}(J') \in J(P)$  whenever  $J' \in J(P')$ . Now, let  $x \in P$ . We have  $\varphi^{-1}(P' \setminus \varphi(x)) := P \setminus \uparrow x$ . Since  $\varphi(x)$  is a finite join of primes,  $P' \setminus \uparrow \varphi(x)$  is a finite union of ideals. Since their inverse images are ideals,  $P \setminus \uparrow x$  is a finite union of ideals too.

(iv)  $\Rightarrow$  (iii) We use the well-known method for representing a poset by a family of sets.

**Fact 1.3.** *Let  $P$  be a poset and  $Q \subseteq I(P)$ . For  $x \in P$  set  $\varphi_Q(x) := \{J \in Q : x \notin J\}$ . Then:*

- (a)  $\varphi_Q(x) \in I(Q)$ ;
- (b)  $\varphi_Q : P \rightarrow I(Q)$  is an order-preserving map;
- (c)  $\varphi_Q$  is an order-embedding if and only if for every  $x, y \in P$  such that  $x \not\leq y$  there is some  $J \in Q$  such that  $x \notin J$  and  $y \in J$ .

Applying this to  $Q := \Delta(J(P))$  we get immediately that  $\varphi_Q$  is join-preserving. Moreover,  $\varphi_Q(x) \in I_{<\omega}(Q)$  if and only if  $P \setminus \uparrow x$  is a finite union of ideals. Indeed, we have  $P \setminus \uparrow x = \cup \varphi_Q(x)$ , proving that  $P \setminus \uparrow x$  is a finite union of ideals provided that  $\varphi_Q(x) \in I_{<\omega}(Q)$ . Conversely, if  $P \setminus \uparrow x$  is a finite union of ideals, say  $I_0, \dots, I_n$ , then since ideals are prime members of  $I(P)$ , every ideal included in  $I$  is included in some  $I_i$ , proving that  $\varphi_Q(x) \in I_{<\omega}(Q)$ . To conclude, note that if  $P$  is a join-semilattice then  $\varphi_Q$  is join-preserving.

(iii)  $\Rightarrow$  (ii) Trivial.

(ii)  $\Rightarrow$  (i) Trivial. □

**Corollary 1.2.** *If a join-semilattice  $P$  has no infinite antichain, it embeds in  $I_{<\omega}(J(P))$  as a join-subsemilattice.*

**Proof.** As is well known, if a poset has no infinite antichain then every initial segment is a finite union of ideals (cf [12], see also [13] 4.7.3 pp. 125). Thus Proposition 1.2 applies. □

Another corollary of Proposition 1.2 is the following.

**Corollary 1.3.** *Let  $P$  be a join-semilattice. If for every  $x \in P$ ,  $P \setminus \uparrow x$  is a finite union of ideals and  $\Delta(J(P))$  is well-founded then  $P$  embeds as a join-subsemilattice in  $I_{<\omega}(Q)$ , for some well-founded poset  $Q$ .*

The converse does not hold:

**Example 1.1.** *There is a bipartite poset  $Q$  such that  $I_{<\omega}(Q)$  contains a join-semilattice  $P$  for which  $\Delta(J(P))$  is not well-founded.*

**Proof.** Let  $\underline{2} := \{0, 1\}$  and  $Q := \mathbb{N} \times \underline{2}$ . Order  $Q$  in such a way that  $(m, i) < (n, j)$  if  $m > n$  in  $\mathbb{N}$  and  $i < j$  in  $\underline{2}$ .

Let  $P$  be the set of subsets  $X$  of  $Q$  of the form  $X := F \times \{0\} \cup G \times \{1\}$  such that  $F$  is a non-empty final segment of  $\mathbb{N}$ ,  $G$  is a non-empty finite subset of  $\mathbb{N}$  and

$$(8) \quad \min(F) - 1 \leq \min(G) \leq \min(F)$$

where  $\min(F)$  and  $\min(G)$  denote the least element of  $F$  and  $G$  w.r.t. the natural order on  $\mathbb{N}$ . For each  $n \in \mathbb{N}$ , let  $I_n := \{X \in P : (n, 0) \notin X\}$ .

**Claim**

- (1)  $Q$  is bipartite and  $P$  is a join-subsemilattice of  $I_{<\omega}(Q)$ .
- (2) The  $I_n$ 's form a strictly descending sequence of members of  $\Delta(J(P))$ .

**Proof of the Claim**

1. The poset  $Q$  is decomposed into two antichains, namely  $\mathbb{N} \times \{0\}$  and  $\mathbb{N} \times \{1\}$  and for this reason is called *bipartite*. Next,  $P$  is a subset of  $I_{<\omega}(Q)$ . Indeed, Let  $X \in P$ . Let  $F, G$  such that  $X = F \times \{0\} \cup G \times \{1\}$ . Set  $G' := G \times \{1\}$ . If  $\min(G) = \min(F) - 1$ , then  $X = \downarrow G'$  whereas if  $\min(G) = \min(F)$  then  $X = \downarrow G' \cup \{(\min(F), 0)\}$ . In both cases  $X \in I_{<\omega}(Q)$ . Finally,  $P$  is a join-semilattice. Indeed, let  $X, X' \in P$  with  $X := F \times \{0\} \cup G \times \{1\}$  and  $X' := F' \times \{0\} \cup G' \times \{1\}$ . Obviously  $X \cup X' = (F \cup F') \times \{0\} \cup (G \cup G') \times \{1\}$ . Since  $X, X' \in P$ ,  $F \cup F'$  is a non-empty final segment of  $\mathbb{N}$  and  $G \cup G'$  is a non-empty finite subset of  $\mathbb{N}$ . We have  $\min(G \cup G') = \min(\{\min(G), \min(G')\}) \leq \min(\{\min(F), \min(F')\}) = \min(F \cup F')$  and similarly  $\min(F \cup F') - 1 = \min\{\min(F), \min(F')\} - 1 = \min\{\min(F) - 1, \min(F') - 1\} \leq \min\{\min(G), \min(G')\} = \min(G \cup G')$ , proving that inequalities as in (8) hold. Thus  $X \cup X' \in I_{<\omega}(Q)$ .

2. Due to its definition,  $I_n$  is a non-empty initial segment of  $P$  which is closed under finite unions, hence  $I_n \in J(P)$ . Let  $X_n := \{(n, 1), (m, 0) : m \geq n + 1\}$  and  $Y_n := X_n \cup \{(n, 0)\}$ . Clearly,  $X_n \in I_n$  and  $Y_n \in P$ . We claim that  $I_n^+ = I_n \vee \{Y_n\}$ .

Indeed, let  $J$  be an ideal containing strictly  $I_n$ . Let  $Y := \{m \in \mathbb{N} : m \geq p\} \times \{0\} \cup G \times \{1\} \in J \setminus I_n$ . Since  $Y \notin I_n$ , we have  $p \leq n$  hence  $Y_n \subseteq Y \cup X_n \in J$ . It follows that  $Y_n \in J$ , thus  $I_n^+ \subseteq J$ , proving our claim. Since  $I_n^+ \neq I_n$ ,  $I_n \in \Delta(J(P))$ . Since, trivially,  $I_n^+ \subseteq I_{n-1}$  we have  $I_n \subset I_{n-1}$ , proving that the  $I_n$ 's form a strictly descending sequence.  $\square$

Let  $E$  be a set and  $\mathcal{F}$  be a subset of  $\mathfrak{P}(E)$ , the power set of  $E$ . For  $x \in E$ , set  $\mathcal{F}_{\neg x} := \{F \in \mathcal{F} : x \notin F\}$  and for  $X \subset \mathcal{F}$ , set  $\overline{X} := \bigcup X$ . Let  $\mathcal{F}^{<\omega}$  (resp.  $\mathcal{F}^\cup$ ) be the collection of finite (resp. arbitrary) unions of members of  $\mathcal{F}$ . Ordered by inclusion,  $\mathcal{F}^\cup$  is a complete lattice, the least element and the largest element being the empty set and  $\bigcup \mathcal{F}$ , respectively.

**Lemma 1.4.** *Let  $Q$  be a poset,  $\mathcal{F}$  be a subset of  $I_{<\omega}(Q)$  and  $P := \mathcal{F}^{<\omega}$  ordered by inclusion.*

- (a) *The map  $X \rightarrow \overline{X}$  is an isomorphism from  $J(P)$  onto  $\mathcal{F}^\cup$  ordered by inclusion.*
- (b) *If  $I \in \Delta(J(P))$  then there is some  $x \in Q$  such that  $I = P_{\neg x}$ .*
- (c) *If  $\downarrow q$  is finite for every  $q \in Q$  then  $\overline{I^+} \setminus \overline{I}$  is finite for every  $I \in J(P)$  and the set  $\varphi_\Delta(X) := \{I \in \Delta(J(P)) : X \not\subseteq I\}$  is finite for every  $X \in P$ .*

**Proof.**

(a) Let  $I$  and  $J$  be two ideals of  $P$ . Then  $J$  contains  $I$  if and only if  $\overline{J}$  contains  $\overline{I}$ . Indeed, if  $I \subseteq J$  then, clearly  $\overline{I} \subseteq \overline{J}$ . Conversely, suppose  $\overline{I} \subseteq \overline{J}$ . If  $X \in I$ , then  $X \subseteq \overline{I}$ , thus  $X \subseteq \overline{J}$ . Since  $X \in I_{<\omega}(Q)$ , and  $X \subseteq \overline{J}$ , there are  $X_1, \dots, X_n \in J$  such that  $X \subseteq Y = X_1 \cup \dots \cup X_n$ . Since  $J$  is an ideal  $Y \in J$ . It follows that  $X \in J$ .

(b) Let  $I \in \Delta(J(P))$ . From (a), we have  $\overline{I} \subset \overline{I^+}$ . Let  $x \in \overline{I^+} \setminus \overline{I}$ . Clearly  $P_{\neg x}$  is an ideal containing  $I$ . Since  $x \notin \overline{P_{\neg x}}$ ,  $P_{\neg x}$  is distinct from  $I^+$ . Hence  $P_{\neg x} = I$ . Note that the converse of assertion (b) does not hold in general.

(c) Let  $I \in \Delta(J(P))$  and  $X \in I^+ \setminus I$ . We have  $\{X\} \vee I = I^+$ , hence from (a)  $\overline{\{X\} \vee I} = \overline{I^+}$ . Since  $\overline{\{X\} \vee I} = X \cup \overline{I}$  we have  $\overline{I^+} \setminus \overline{I} \subseteq X$ . From our hypothesis on  $P$ ,  $X$  is finite, hence  $\overline{I^+} \setminus \overline{I}$  is finite. Let  $X \in P$ . If  $I \in \varphi_\Delta(X)$  then according to (b) there is some  $x \in Q$  such that  $I = P_{\neg x}$ . Necessarily  $x \in X$ . Since  $X$  is finite, the number of these  $I$ 's is finite.  $\square$

**Proposition 1.3.** *Let  $P$  be a join-semilattice. The following properties are equivalent:*

- (i)  *$P$  embeds in  $[E]^{<\omega}$  as a join-subsemilattice for some set  $E$ ;*
- (ii) *for every  $x \in P$ ,  $\varphi_\Delta(x)$  is finite.*

**Proof.** (i)  $\Rightarrow$  (ii) Let  $\varphi$  be an embedding from  $P$  in  $[E]^{<\omega}$  which preserves joins. Set  $\mathcal{F} := \varphi(P)$ . Apply part (c) of Lemma 1.4. (ii)  $\Rightarrow$  (i) Set  $E := \Delta(J(P))$ . We have  $\varphi_\Delta(x) \in [E]^{<\omega}$ . According to Fact 1.3 and Lemma 3.6, the map  $\varphi_\Delta : P \rightarrow [E]^{<\omega}$  is an embedding preserving joins.  $\square$

**Corollary 1.4.** *Let  $\beta$  be a countable order type. If a proper initial segment contains infinitely many non-principal initial segments then no sierpinskiation  $P$  of  $\beta$  with  $\omega$  can embed in  $[\omega]^{<\omega}$  as a join-semilattice (whereas it embeds as a poset).*

**Proof.** According to Proposition 1.3 it suffices to prove that  $P$  contains some  $x$  for which  $\varphi_\Delta(x)$  is infinite.

Let  $P$  be a sierpinskiisation of  $\beta$  and  $\omega$ . It is obtained as the intersection of two linear orders  $L, L'$  on the same set and having respectively order type  $\beta$  and  $\omega$ . We may suppose that the ground set is  $\mathbb{N}$  and  $L'$  the natural order.

**Claim 1** A non-empty subset  $I$  is a non-principal ideal of  $P$  if and only if this is a non-principal initial segment of  $L$ .

**Proof of Claim 1** Suppose that  $I$  is a non-principal initial segment of  $L$ . Then, clearly,  $I$  is an initial segment of  $P$ . Let us check that  $I$  is up-directed. Let  $x, y \in I$ ; since  $I$  is non-principal in  $L$ , the set  $A := I \cap \uparrow_L x \cap \uparrow_L y$  of upper-bounds of  $x$  and  $y$  w.r.t.  $L$  which belong to  $I$  is infinite; since  $B := \downarrow_{L'} x \cup \downarrow_{L'} y$  is finite,  $A \setminus B$  is non-empty. An arbitrary element  $z \in A \setminus B$  is an upper bound of  $x, y$  in  $I$  w.r.t. the poset  $P$  proving that  $I$  is up-directed. Since  $I$  is infinite,  $I$  cannot have a largest element in  $P$ , hence  $I$  is a non-principal ideal of  $P$ . Conversely, suppose that  $I$  is a non-principal ideal of  $P$ . Let us check that  $I$  is an initial segment of  $L$ . Let  $x \leq_L y$  with  $y \in I$ . Since  $I$  non-principal in  $P$ ,  $A := \uparrow_P y \cap I$  is infinite; since  $B := \downarrow_{L'} x \cup \downarrow_{L'} y$  is finite,  $A \setminus B$  is non-empty. An arbitrary element of  $A \setminus B$  is an upper bound of  $x$  and  $y$  in  $I$  w.r.t.  $P$ . It follows that  $x \in I$ . If  $I$  has a largest element w.r.t.  $L$  then such an element must be maximal in  $I$  w.r.t.  $P$ , and since  $I$  is an ideal,  $I$  is a principal ideal, a contradiction.

**Claim 2** Let  $x \in \mathbb{N}$ . If there is a non-principal ideal of  $L$  which does not contain  $x$ , there is a maximal one, say  $I_x$ . If  $P$  is a join-semilattice,  $I_x \in \Delta(P)$ .

**Proof of Claim 2** The first part follows from Zorn's Lemma. The second part follows from Claim 1 and Lemma 3.6.

**Claim 3** If an initial segment  $I$  of  $\beta$  contains infinitely many non-principal initial segments then there is an infinite sequence  $(x_n)_{n < \omega}$  of elements of  $I$  such that the  $I_{x_n}$ 's are all distinct.

**Proof of Claim 3** With Ramsey's theorem obtain a sequence  $(I_n)_{n < \omega}$  of non-principal initial segments which is either strictly increasing or strictly decreasing. Separate two successive members by some element  $x_n$  and apply the first part of Claim 2.

If we pick  $x \in \mathbb{N} \setminus I$  then it follows from Claim 3 and the second part of Claim 2 that  $\varphi_\Delta(x)$  is infinite.  $\square$

**Example 1.2.** If  $\alpha$  is a countably infinite order type distinct from  $\omega$ ,  $\Omega(\alpha)$  is not embeddable in  $[\omega]^{<\omega}$  as a join-semilattice.

Indeed,  $\Omega(\alpha)$  is a sierpinskiisation of  $\omega\alpha$  and  $\omega$ . And if  $\alpha$  is distinct from  $\omega$ ,  $\alpha$  contains some element which majorizes infinitely many others. Thus  $\beta := \omega\alpha$  satisfies the hypothesis of Corollary 1.4.

Note that on an other hand, for every ordinal  $\alpha \leq \omega$ , there are representatives of  $\Omega(\alpha)$  which are embeddable in  $[\omega]^{<\omega}$  as join-semilattices.

**Theorem 1.7.** Let  $Q$  be a well-founded poset and let  $\mathcal{F} \subseteq I_{<\omega}(Q)$ . The following properties are equivalent:

- 1)  $\mathcal{F}$  has no infinite antichain;
- 2)  $\mathcal{F}^{<\omega}$  is wqo;
- 3)  $J(\mathcal{F}^{<\omega})$  is topologically scattered;



- 4)  $\mathcal{F}^\cup$  is order-scattered;
- 5)  $\mathfrak{P}(\omega)$  does not embed in  $\mathcal{F}^\cup$ ;
- 6)  $[\omega]^{<\omega}$  does not embed in  $\mathcal{F}^{<\omega}$ ;
- 7)  $\mathcal{F}^\cup$  is well-founded.

**Proof.** We prove the following chain of implications:

$$1) \implies 2) \implies 3) \implies 4) \implies 5) \implies 6) \implies 7) \implies 1)$$

1)  $\implies$  2). Since  $Q$  is well-founded then, as mentioned in a) of Theorem 1.6,  $I_{<\omega}(Q)$  is well-founded. It follows first that  $\mathcal{F}^{<\omega}$  is well-founded, hence from Property c) of Theorem 1.6, every member of  $\mathcal{F}^{<\omega}$  is a finite join of join-irreducibles. Next, as a subset of  $\mathcal{F}^{<\omega}$ ,  $\mathcal{F}$  is well-founded, hence wqo according to our hypothesis. The set of join-irreducible members of  $\mathcal{F}^{<\omega}$  is wqo as a subset of  $\mathcal{F}$ . From Property d) of Theorem 1.6,  $\mathcal{F}^{<\omega}$  is wqo

2)  $\implies$  3). If  $\mathcal{F}^{<\omega}$  is wqo then  $I(\mathcal{F}^{<\omega})$  is well-founded (cf. Property (b) of Theorem 1.6). It follows that  $I(\mathcal{F}^{<\omega})$  is topologically scattered (cf. [28]); hence all its subsets are topologically scattered, in particular  $J(\mathcal{F}^{<\omega})$ .

3)  $\implies$  4). Suppose that  $\mathcal{F}^\cup$  is not ordered scattered. Let  $f : \eta \rightarrow \mathcal{F}^\cup$  be an embedding. For  $r \in \eta$  set  $\check{f}(r) = \bigcup \{f(r') : r' < r\}$ . Let  $X := \{\check{f}(r) : r < \eta\}$ . Clearly  $X \subseteq \mathcal{F}^\cup$ . Furthermore  $X$  contains no isolated point (Indeed, since  $\check{f}(r) = \bigcup \{\check{f}(r') : r' < r\}$ ,  $\check{f}(r)$  belongs to the topological closure of  $\{\check{f}(r') : r' < r\}$ ). Hence  $\mathcal{F}^\cup$  is not topologically scattered.

4)  $\implies$  5). Suppose that  $\mathfrak{P}(\omega)$  embeds in  $\mathcal{F}^\cup$ . Since  $\eta \leq \mathfrak{P}(\omega)$ , we have  $\eta \leq \mathcal{F}^\cup$ .

5)  $\implies$  6). Suppose that  $[\omega]^{<\omega}$  embeds in  $\mathcal{F}^{<\omega}$ , then  $J([\omega]^{<\omega})$  embeds in  $J(\mathcal{F}^{<\omega})$ . Lemma 1.4 assures that  $J(\mathcal{F}^{<\omega})$  is isomorphic to  $\mathcal{F}^\cup$ . In the other hand  $J([\omega]^{<\omega})$  is isomorphic to  $\mathfrak{P}(\omega)$ . Hence  $\mathfrak{P}(\omega)$  embeds in  $\mathcal{F}^\cup$ .

6)  $\implies$  7). Suppose  $\mathcal{F}^\cup$  not well-founded. Since  $Q$  is well-founded, a) of Theorem 1.6 assures  $I_{<\omega}(Q)$  well-founded, but  $\mathcal{F}^{<\omega} \subseteq I_{<\omega}(Q)$ , hence  $\mathcal{F}^{<\omega}$  is well-founded. Furthermore, since  $I_{<\omega}(Q)$  is closed under finite unions, we have  $\mathcal{F}^{<\omega} \subseteq I_{<\omega}(Q)$ , Proposition 2.1 implies that  $\underline{\Omega}(\omega^*)$  does not embed in  $\mathcal{F}^{<\omega}$ . From Theorem 1.2, we have  $\mathcal{F}^{<\omega}$  not well-founded.

7)  $\implies$  1). Clearly,  $\mathcal{F}$  is well-founded. If  $F_0, \dots, F_n, \dots$  is an infinite antichain of members of  $\mathcal{F}$ , define  $f(i, j) : [\omega]^2 \rightarrow Q$ , choosing  $f(i, j)$  arbitrary in  $\text{Max}(F_i) \setminus F_j$ . Divide  $[\omega]^3$  into  $R_1 := \{(i, j, k) \in [\omega]^3 : f(i, j) = f(i, k)\}$  and  $R_2 := [\omega]^3 \setminus R_1$ . From Ramsey's theorem, cf. [37], there is some infinite subset  $X$  of  $\omega$  such that  $[X]^3$  is included in  $R_1$  or in  $R_2$ . The inclusion in  $R_2$  is impossible since  $\{f(i, j) : j < \omega\}$ , being included in  $\text{Max}(F_i)$ , is finite for every  $i$ . For each  $i \in X$ , set  $G_i := \bigcup \{F_j : i \leq j \in X\}$ . This defines an  $\omega^*$ -chain in  $\mathcal{F}^\cup$ .  $\square$

**Remark 1.2.** If  $\mathcal{F}^{<\omega}$  is closed under finite intersections then equivalence between (3) and (4) follows from Mislove's Theorem mentioned in [28].

Theorem 1.7 above was obtained by the second author and M.Sobrani in the special case where  $Q$  is an antichain [33, 40].

**Corollary 1.5.** *If  $P$  is a join-subsemilattice of a join-semilattice of the form  $[\omega]^{<\omega}$ , or more generally of the form  $I_{<\omega}(Q)$  where  $Q$  is some well-founded poset, then  $J(P)$  is well-founded if and only if  $P$  has no infinite antichain.*

**Remark.** If, in Theorem 1.7 above, we suppose that  $\mathcal{F}$  is well-founded instead of  $Q$ , all implications in the above chain hold, except  $6) \Rightarrow 7)$ . A counterexample is provided by  $Q := \omega \oplus \omega^*$ , the direct sum of the chains  $\omega$  and  $\omega^*$ , and  $\mathcal{F}$ , the image of  $\underline{\Omega}(\omega^*)$  via a natural embedding.

**4.1. Proof of Theorem 1.5.**  $(i) \Rightarrow (ii)$  Suppose that  $(i)$  holds. Set  $Q := J(P)$ . Since  $P$  contains no infinite antichain,  $P$  embeds as a join-subsemilattice in  $I_{<\omega}(Q)$  (Corollary 1.2). From  $b)$  of Theorem 1.6  $Q$  is well-founded. Since  $P$  has no infinite antichain, it has no infinite independent set.

$(ii) \Rightarrow (i)$  Suppose that  $(ii)$  holds. Since  $Q$  is well-founded, then from  $a)$  of Theorem 1.6,  $I_{<\omega}(Q)$  is well-founded. Since  $P$  embeds in  $I_{<\omega}(Q)$ ,  $P$  is well-founded. From our hypothesis,  $P$  contains no infinite independent set. According to implication  $(iii) \Rightarrow (i)$  of Theorem 1.3, it does not embed  $[\omega]^{<\omega}$ . From implication  $6) \Rightarrow 1)$  of Theorem 1.7, it has no infinite antichain.  $\square$

## CHAPTER 2

### On the length of chains in algebraic lattices

1

We study how the existence of a chain of a given type in an algebraic lattice  $L$  is reflected in the join-semilattice  $K(L)$  of its compact elements. We show that for every chain  $\alpha$  of size  $\kappa$ , there is a set  $\mathbb{B}$  of at most  $2^\kappa$  join-semilattices, each one having a least element such that an algebraic lattice  $L$  contains no chain of order type  $I(\alpha)$  if and only if the join-semilattice  $K(L)$  of its compact elements contains no join-subsemilattice isomorphic to a member of  $\mathbb{B}$ . We show that among the join-subsemilattices of  $[\omega]^{<\omega}$  belonging to  $\mathbb{B}$ , one is embeddable in all the others. We conjecture that if  $\alpha$  is countable, there is a finite  $\mathbb{B}$ . We study some special cases, particularly when  $\alpha$  is an ordinal.

#### 1. Introduction

This paper is about the relationship between the length of chains in an algebraic lattice  $L$  and the structure of the join-semilattice  $K(L)$  of the compact elements of  $L$ . We started such an investigation in [6], [7], [8], [9]. We present first the motivation.

Let  $P$  be an ordered set (poset). An *ideal* of  $P$  is any non-empty up-directed initial segment of  $P$ . The set  $J(P)$  of ideals of  $P$ , ordered by inclusion, is an interesting poset associated with  $P$ . For a concrete example, if  $P := [\kappa]^{<\omega}$  the set, ordered by inclusion, consisting of finite subsets of a set of size  $\kappa$ , then  $J([\kappa]^{<\omega})$  is isomorphic to  $\mathfrak{P}(\kappa)$  the power set of  $\kappa$  ordered by inclusion. In [8] we proved:

**Theorem 2.1.** *A poset  $P$  contains a subset isomorphic to  $[\kappa]^{<\omega}$  if and only if  $J(P)$  contains a subset isomorphic to  $\mathfrak{P}(\kappa)$ .*

Maximal chains in  $\mathfrak{P}(\kappa)$  are of the form  $I(C)$ , where  $I(C)$  is the chain of initial segments of an arbitrary chain  $C$  of size  $\kappa$  (cf. [4]). Hence, if  $J(P)$  contains a subset isomorphic to  $\mathfrak{P}(\kappa)$  it contains a copy of  $I(C)$  for every chain  $C$  of size  $\kappa$ , whereas chains in  $P$  can be small: eg in  $P := [\kappa]^{<\omega}$  they are finite or have order type  $\omega$ . What happens if for a given order type  $\alpha$ , particularly a countable one,  $J(P)$  contains no chain of type  $\alpha$ ? A partial answer was given by Pouzet, Zaguia, 1984 (cf. [34] Theorem 4, pp.62). In order to state their result, we recall that the order type  $\alpha$  of a chain  $C$  is *indecomposable* if  $C$  can be embedded in each non-empty final segment of  $C$ .

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<sup>1</sup>Les principaux résultats de ce chapitre sont inclus dans l'article: I.Chakir, M.Pouzet, The length of chains in algebraic lattices, Les annales ROAD du LAID3, special issue 2008, pp 379-390 (proceedings of ISOR'08, Algiers, Algeria, Nov 2-6, 2008).

**Theorem 2.2.** <sup>2</sup> *Given an indecomposable countable order type  $\alpha$ , there is a finite list of ordered sets  $A_1^\alpha, A_2^\alpha, \dots, A_{n_\alpha}^\alpha$  such that for every poset  $P$ , the set  $J(P)$  of ideals of  $P$  contains no chain of type  $I(\alpha)$  if and only if  $P$  contains no subset isomorphic to one of the  $A_1^\alpha, A_2^\alpha, \dots, A_{n_\alpha}^\alpha$ .*

If  $P$  is a join-semilattice with a least element,  $J(P)$  is an algebraic lattice, moreover every algebraic lattice is isomorphic to the poset  $J(K(L))$  of ideals of the join-semilattice  $K(L)$  of the compact elements of  $L$  (see [15]). It is natural to ask whether the two results above change if the poset  $P$  is a join-semilattice and if one consider join-subsemilattices instead of subsets of  $P$ .

The specialization of Theorem 2.1 to this case is immediate (in fact easier to prove) and shows no difference. Indeed a join-semilattice  $P$  contains a subset isomorphic to  $[\kappa]^{<\omega}$  if and only if it contains a join-subsemilattice isomorphic to  $[\kappa]^{<\omega}$ . The specialization of Theorem 2.2 turns to be different and is far from being immediate. In fact, we do not know yet whether there is a finite list.

At first glance, if the set  $J(P)$  of ideals of a join-semilattice  $P$  contains a chain of type  $I(\alpha)$  then according to Theorem 2.2,  $P$  contains, as a poset, one of the  $A_i^\alpha$ 's. Thus,  $P$  contains, as a join-subsemilattice, the join-semilattice generated by  $A_i^\alpha$  in  $P$ . A description of the join-semilattices generated by the  $A_i^\alpha$ 's would lead to the specialization of Theorem 2.1. We have been unable to succeed in this direction. The only results we have obtained so far have been obtained by mimicking the proof of Theorem 2.2.

In this result, the  $A_1^\alpha, \dots, A_{n_\alpha}^\alpha$ 's are the "obstructions" to the existence of a chain of type  $\alpha$ . Typical obstructions are built via sierpinskiations. Let  $\alpha$  be a countable chain and  $\omega$  be the chain of non-negative integers. A *sierpinskiation* of  $\alpha$  and  $\omega$ , or simply of  $\alpha$ , is any poset  $(S, \leq)$  such that the order on  $S$  is the intersection of two linear orders on  $S$ , one of type  $\alpha$ , the other of type  $\omega$ . Such a sierpinskiation can be obtained from a bijective map  $\varphi : \omega \rightarrow \alpha$ , setting  $S := \mathbb{N}$  and  $x \leq y$  if  $x \leq y$  w.r.t. the natural order on  $\mathbb{N}$  and  $\varphi(x) \leq \varphi(y)$  w.r.t. the order of type  $\alpha$ . The proof of Theorem 2.2 involves sierpinskiations of  $\omega \cdot \alpha$  and  $\omega$ , where  $\omega \cdot \alpha$  is the ordinal sum of  $\alpha$  copies of the chain  $\omega$ , these sierpinskiations being obtained from bijective maps  $\varphi : \omega \rightarrow \omega \cdot \alpha$  such that  $\varphi^{-1}$  is order-preserving on each subset of the form  $\omega \times \{\beta\}$  where  $\beta \in \alpha$ . For brevity, we say that these sierpinskiations are *monotonic*. Augmented of a least element, if it has none, a monotonic sierpinskiation contains a chain of ideals of type  $I(\alpha)$ . Moreover, all posets obtained via this process can be embedded in each other. They are denoted by the same symbol  $\underline{\Omega}(\alpha)$  (cf. [34] Lemma 3.4.3). If  $\alpha = \omega^*$  or  $\alpha = \eta$ , the list is reduced to  $\underline{\Omega}(\alpha)$  and  $\alpha$ . For other order types, there are other obstructions. They are obtained by means of lexicographical sums of obstructions corresponding to chains of order type strictly less than  $\alpha$ .

As we will see, among the monotonic sierpinskiations of  $\omega \cdot \alpha$  and  $\omega$  there are some which are join-subsemilattices of the direct product  $\omega \times \alpha$  that we call *lattice*

<sup>2</sup>In Theorem 4,  $I(\alpha)$  is replaced by  $\alpha$ . This is due to the fact that if  $\alpha$  is a countable indecomposable order type and  $P$  is a poset,  $I(\alpha)$  can be embedded into  $J(P)$  if and only if  $\alpha$  can be embedded into  $J(P)$ .

*sierpinskiations*. This suggests to prove the specialization of Theorem 2.2 along the same lines. We succeeded for  $\alpha = \omega^*$ . We did not for  $\alpha = \eta$ .

We observe that for other countable chains, there are other obstructions that we have to take into account:

In Theorem 2.2,  $[\omega]^{<\omega}$  never occurs in the list  $A_1^\alpha, A_2^\alpha, \dots, A_{n_\alpha}^\alpha$ . We will prove in this paper that  $[\omega]^{<\omega}$  occurs necessarily in a list if and only if  $\alpha$  is not an ordinal (Theorem 2.5). And we will prove that if  $\alpha$  is an ordinal then  $[\omega]^{<\omega}$  contains an obstruction which necessarily occurs in every list of obstructions. This obstruction is the join-semilattice  $Q_\alpha := I_{<\omega}(S_\alpha)$  made of the finitely generated initial segments of  $S_\alpha$ , where  $S_\alpha$  is a sierpinskiation of  $\alpha$  and  $\omega$  (Theorem 2.6).

We conjecture that with these extra obstructions added, the specialization of Theorem 2.2 can be obtained. We guess that the case of ordinal number is not far away. But we are only able to give an answer in very few cases.

## 2. Presentation of the results

Let  $\mathbb{A}$ , resp.  $\mathbb{J}$ , be the class of algebraic lattices, resp. join-semilattices having a least element. Given an order type  $\alpha$ , let  $I(\alpha)$  be the order type of the chain  $I(C)$  of initial segments of a chain  $C$  of order type  $\alpha$ , let  $\mathbb{A}_{-\alpha}$  be the class of algebraic lattices  $L$  such that  $L$  contains no chain of order type  $I(\alpha)$ , let  $\mathbb{J}_{-\alpha}$  be the subclass of  $P \in \mathbb{J}$  such that  $J(P) \in \mathbb{A}_{-\alpha}$ , let  $\mathbb{J}_\alpha := \mathbb{J} \setminus \mathbb{J}_{-\alpha}$  and, for a subcollection  $\mathbb{B}$  of  $\mathbb{J}$ , let  $Forb_{\mathbb{J}}(\mathbb{B})$  be the class of  $P \in \mathbb{J}$  such that no member of  $\mathbb{B}$  is isomorphic to a join-subsemilattice of  $P$ . We ask:

**Question 2.1.** *Find  $\mathbb{B}$  as simple as possible such that:*

$$(9) \quad L \in \mathbb{A}_{-\alpha} \text{ if and only if } K(L) \in Forb_{\mathbb{J}}(\mathbb{B}).$$

*or equivalently:*

$$(10) \quad \mathbb{J}_{-\alpha} = Forb_{\mathbb{J}}(\mathbb{B}).$$

We prove that we can find some  $\mathbb{B}$  of size at most  $2^{|\alpha|}$ .

**Theorem 2.3.** *Let  $\alpha$  be an order type. There is a list  $\mathbb{B}$  of join-semilattices, of size at most  $2^{|\alpha|}$ , such that for every join-semilattice  $P$ , the lattice  $J(P)$  of ideals of  $P$  contains no chain of order type  $I(\alpha)$  if and only if  $P$  contains no join-subsemilattice isomorphic to a member of  $\mathbb{B}$ .*

This is very weak. Indeed, we cannot answer the following question.

**Question 2.2.** *If  $\alpha$  is countable, does equation (10) holds for some finite subset  $\mathbb{B}$  of  $\mathbb{J}$ ?*

Questions and results above can be recast in terms of a quasi-order. Let  $P, P' \in \mathbb{J}$ , set  $P \leq P'$  if  $P'$  is isomorphic to a join-subsemilattice of  $P$ . This relation is a quasi-order on  $\mathbb{J}$ . If  $\alpha$  is an order type,  $\mathbb{J}_{-\alpha}$  is an initial segment of  $\mathbb{J}$ , that is  $P' \in \mathbb{J}_{-\alpha}$  and  $P \leq P'$  imply  $P \in \mathbb{J}_{-\alpha}$ . Indeed, from  $P \leq P'$  we get an embedding from  $J(P)$  into  $J(P')$  which preserves arbitrary joins. If  $I(\alpha)$  was embeddable in  $J(P)$  it would be embeddable in  $J(P')$ , which is not the case. Hence,  $P \in \mathbb{J}_{-\alpha}$ .

A class  $\mathbb{B}$  satisfying (9) is *coinitial* in  $\mathbb{J}_\alpha$ , in the sense that for every  $P' \in \mathbb{J}_\alpha$  there is some  $P \in \mathbb{B}$  such that  $P' \leq P$ .

The existence of a finite cofinal  $\mathbb{B}$  amounts to the fact that, w.r.t. the order on the quotient,  $\mathbb{J}_\alpha$  has finitely many minimal elements and every element of  $\mathbb{J}_\alpha$  is above some. Thus, as far as we identify two join-semilattices which are embeddable in each other as join-semilattices

**Lemma 2.1.**  $\mathbb{J}_\alpha$  contains  $1 + \alpha$  and  $[E]^{<\omega}$ , where  $E$  is the domain of the chain  $\alpha$ .

**Proof.** Since  $J(1 + \alpha) = 1 + J(\alpha) = I(\alpha)$ ,  $1 + \alpha \in \mathbb{J}_\alpha$ . As mentioned above,  $J([E]^{<\omega})$  is isomorphic to  $\mathfrak{P}(E)$  and since  $\alpha$  is a linear order on  $E$ ,  $I(\alpha)$  is isomorphic to a maximal chain of  $[E]^{<\omega}$ , hence  $[E]^{<\omega} \in \mathbb{J}_\alpha$ .  $\square$

As one can immediately see:

**Lemma 2.2.**  $1 + \alpha$  belongs to every  $\mathbb{B}$  coinitial in  $\mathbb{J}_\alpha$ .

**Proof.** In terms of the quasi-order, this assertion amounts to the fact that  $1 + \alpha$  is minimal in  $\mathbb{J}_\alpha$ . As shown in Lemma 2.1,  $1 + \alpha \in \mathbb{J}_\alpha$ . If  $Q \in \mathbb{J}_\alpha$  and  $Q \leq 1 + \alpha$  then since  $Q$  has a least element, we have  $Q = 1 + \beta$  with  $\beta \leq \alpha$ . Since  $J(Q) = I(\beta)$ , from  $Q \in \mathbb{J}_\alpha$ , we get  $I(\alpha) \leq I(\beta)$ . This implies  $\alpha \leq \beta$  and  $1 + \alpha \leq Q$ . Hence  $1 + \alpha$  is minimal in  $\mathbb{J}_\alpha$  as claimed.  $\square$

If  $\alpha$  is a finite chain, or the chain  $\omega$  of non-negative integers, one can easily see that  $1 + \alpha$  is the least element of  $\mathbb{J}_\alpha$ . Thus, one can take  $\mathbb{B} = \{1 + \alpha\}$ .

Sierpinskiations come in the picture:

**Lemma 2.3.** If  $\alpha$  is a countably infinite order type and  $S$  is a sierpinskiation of  $\alpha$  and  $\omega$  then the join-semilattice  $I_{<\omega}(S)$ , made of finitely generated initial segments of  $S$ , is isomorphic to a join-subsemilattice of  $[\omega]^{<\omega}$  and belongs to  $\mathbb{J}_\alpha$ .

**Proof.** By definition, the order on a sierpinskiation  $S$  of  $\alpha$  and  $\omega$  has a linear extension such that the resulting chain  $\bar{S}$  has order type  $\alpha$ . The chain  $I(\bar{S})$  is a maximal chain of  $I(S)$  of type  $I(\alpha)$ . The lattices  $I(S)$  and  $J(I_{<\omega}(S))$  are isomorphic, thus  $I_{<\omega}(S) \in \mathbb{J}_\alpha$ . The order on  $S$  has a linear extension of type  $\omega$ , thus every principal initial segment of  $S$  is finite and more generally every finitely generated initial segment of  $S$  is finite. This tells us that  $I_{<\omega}(S)$  is a join-subsemilattice of  $[S]^{<\omega}$ . Since  $S$  is countable,  $I_{<\omega}(S)$  identifies to a join-subsemilattice of  $[\omega]^{<\omega}$ .  $\square$

**Remark 2.1.** If  $\alpha$  is not an ordinal, Lemma 2.3 tells us nothing new. Indeed, in this case any sierpinskiation  $S$  of  $\alpha$  and  $\omega$  contains an infinite antichain, hence  $I_{<\omega}(S)$  and  $[\omega]^{<\omega}$  are embeddable in each other as join-semilattices.

There is a much deeper result:

**Theorem 2.4.** If  $\alpha$  is a countable order type then among the join-subsemilattices  $P$  of  $[\omega]^{<\omega}$  which belong to  $\mathbb{J}_\alpha$  there is one which embeds as a join-semilattice in all the others. This join-semilattice is of the form  $I_{<\omega}(S_\alpha)$  where  $S_\alpha$  is a sierpinskiation of  $\alpha$  and  $\omega$ .

We deduce it from Theorem 2.5 and Theorem 2.6 below:

**Theorem 2.5.** *Let  $\alpha$  be a countable order type. The join-semilattice  $[\omega]^{<\omega}$  belongs to every  $\mathbb{B}$  cointial in  $\mathbb{J}_\alpha$  if and only if  $\alpha$  is not an ordinal.*

Let  $\alpha$  be an ordinal. Set  $S_\alpha := \alpha$  if  $\alpha < \omega$ . If  $\alpha = \omega\alpha' + n$  with  $\alpha' \neq 0$  and  $n < \omega$ , let  $S_\alpha := \Omega(\alpha') \oplus n$  be the direct sum of  $\Omega(\alpha')$  and the chain  $n$ , where  $\Omega(\alpha')$  is a monotonic sierpinskisation of  $\omega\alpha'$  and  $\omega$ . We note that for countably infinite  $\alpha$ 's,  $S_\alpha$  is a sierpinskisation of  $\alpha$  and  $\omega$ . We prove that  $Q_\alpha := I_{<\omega}(S_\alpha)$  has the property stated in Theorem 2.4:

**Theorem 2.6.** *If  $\alpha$  is an ordinal then  $I_{<\omega}(S_\alpha)$  is a join-subsemilattice of  $[\omega]^{<\omega}$  which belongs to  $\mathbb{J}_\alpha$  and is embeddable as a join-semilattice in all join-subsemilattices of  $[\omega]^{<\omega}$  which belongs to  $\mathbb{J}_\alpha$*

The deduction of Theorem 2.4 from these two results is immediate:

If  $\alpha$  is an ordinal, apply Theorem 2.6. If  $\alpha$  is not an ordinal, then according to Remark 2.1 above, the conclusion of Theorem 2.4 amounts to the fact that  $[\omega]^{<\omega}$  is minimal in  $\mathbb{J}_\alpha$ .

The proof of Theorem 2.6 is given in Section 4. We discuss here the proof of Theorem 2.5.

The "only if" part of Theorem 2.5 is easy. It follows from Lemma 2.3 and the following:

**Lemma 2.4.** *If  $\alpha$  is an ordinal and  $S$  is a sierpinskisation of  $\alpha$  and  $\omega$ , then  $[\omega]^{<\omega}$  is not embeddable in  $I_{<\omega}(S)$ .*

This simple fact relies on the important notion of well-quasi-ordering introduced by Higman [17]. We recall that a poset  $P$  is *well-quasi-ordered* (briefly w.q.o.) if every non-empty subset  $A$  of  $P$  has at least a minimal element and the number of these minimal elements is finite. As shown by Higman, this is equivalent to the fact that  $I(P)$  is well-founded [17].

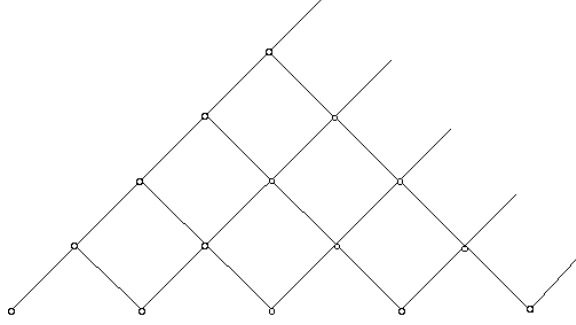
Well-ordered set are trivially w.q.o. and, as it is well known, the direct product of finitely many w.q.o. is w.q.o. Lemma 2.4 follows immediately from this. Indeed, if  $S$  is a sierpinskisation of  $\alpha$  and  $\omega$ , it embeds in the direct product  $\omega \times \alpha$ . Thus  $S$  is w.q.o. and consequently  $I(S)$  is well-founded. This implies that  $[\omega]^{<\omega}$  is not embeddable in  $I_{<\omega}(S)$ . Otherwise  $J([\omega]^{<\omega})$  would be embeddable in  $J(I_{<\omega}(S))$ , that is  $\mathfrak{P}(\omega)$  would be embeddable in  $I(S)$ . Since  $\mathfrak{P}(\omega)$  is not well-founded, this would contradict the well-foundedness of  $I(S)$ .

The "if" part is based on our earlier work on well-founded algebraic lattices, that we record here.

Let  $\omega^*$  be the chain of negative integers. Let  $\underline{\Omega}(\omega^*)$  be the join-semilattice obtained by adding a least element to the set  $[\omega]^2$  of two-element subsets of  $\omega$ , identified to pairs  $(i, j)$ ,  $i < j < \omega$ , ordered so that  $(i, j) \leq (i', j')$  if and only if  $i' \leq i$  and  $j \leq j'$ .

We claim that if  $\alpha = \omega^*$ ,  $\mathbb{J}_\alpha$  has a cointial set made of three join-semilattices, namely  $\omega^*$ ,  $[\omega]^{<\omega}$  and  $\underline{\Omega}(\omega^*)$ . Alternatively

$$(11) \quad \mathbb{J}_{-\omega^*} = \text{Forb}_{\mathbb{J}}(\{1 + \omega^*, \underline{\Omega}(\omega^*), [\omega]^{<\omega}\}).$$

FIGURE 2.1.  $\underline{\Omega}(\omega^*)$ 

This fact is not straightforward. Reformulated in simpler terms, as follows, this is the main result of [9] (cf. Chapter 1, Theorem 1.2).

**Theorem 2.7.** *An algebraic lattice  $L$  is well-founded if and only if  $K(L)$  is well-founded and contains no join-subsemilattice isomorphic to  $\underline{\Omega}(\omega^*)$  or to  $[\omega]^{<\omega}$ .*

From Theorem 2.7, we obtained:

**Theorem 2.8.** (Corollary 1.1, [9]) *A join-subsemilattice  $P$  of  $[\omega]^{<\omega}$  contains either  $[\omega]^{<\omega}$  as a join-semilattice or is well-quasi-ordered. In the latter case,  $J(P)$  is well-founded.*

With this result, the proof of the “if” part of Theorem 2.5 is immediate. Indeed, suppose that  $\alpha$  is not an ordinal. Let  $P \in \mathbb{J}_\alpha$ . The lattice  $J(P)$  contains a chain isomorphic to  $I(\alpha)$ . Since  $\alpha$  is not an ordinal,  $\omega^* \leq \alpha$ . Hence,  $J(P)$  is not well-founded. If  $P$  is embeddable in  $[\omega]^{<\omega}$  as a join-semilattice then, from Theorem 2.8,  $P$  contains a join-subsemilattice isomorphic to  $[\omega]^{<\omega}$ . Thus  $[\omega]^{<\omega}$  is minimal in  $\mathbb{J}_\alpha$ .

A sierpinskiisation  $S$  of a countable order type  $\alpha$  and  $\omega$  is embeddable into  $[\omega]^{<\omega}$  as a poset. A consequence of Theorem 2.8 is the following

**Lemma 2.5.** *If  $S$  can be embedded in  $[\omega]^{<\omega}$  as a join-semilattice,  $\alpha$  must be an ordinal.*

**Proof.** Otherwise,  $S$  contains an infinite antichain and by Theorem 2.8 it contains a copy of  $[\omega]^{<\omega}$ . But this poset cannot be embedded in a sierpinskiisation. Indeed, a sierpinskiisation is embeddable into a product of two chains, whereas  $[\omega]^{<\omega}$  cannot be embedded in a product of finitely many chains (for every integer  $n$ , it contains the power set  $\mathfrak{P}(\{0, \dots, n-1\})$  which cannot be embedded into a product of less than  $n$  chains; its dimension, in the sense of Dushnik-Miller’s notion of dimension, is infinite, see [41])  $\square$ .

This fact invited us to restrict our notion of sierpinskiisation to some which are join-semilattices. In fact, there are lattices of a particular kind.

To a countable order type  $\alpha$ , we associate a join-subsemilattice  $\Omega_L(\alpha)$  of the direct product  $\omega \times \alpha$  obtained via a sierpinskiisation of  $\omega\alpha$  and  $\omega$ . We add a least element, if there is none, and we denote by  $\underline{\Omega}_L(\alpha)$  the resulting poset. We also associate the join-semilattice  $P_\alpha$  defined as follows:



If  $1+\alpha \not\leq \alpha$ , in which case  $\alpha = n+\alpha'$  with  $n < \omega$  and  $\alpha'$  without a first element, we set  $P_\alpha := n + \underline{\Omega}_L(\alpha')$ . If not, and  $\alpha$  is equimorphic to  $\omega + \alpha'$  we set  $P_\alpha := \underline{\Omega}_L(1 + \alpha')$ , otherwise, we set  $P_\alpha = \underline{\Omega}_L(\alpha)$ .

The importance of this kind of sierpinskisation stems from the following result:

**Theorem 2.9.** *If  $\alpha$  is countably infinite,  $P_\alpha$  belongs to every  $\mathbb{B}$  coinital in  $\mathbb{J}_\alpha$ .*

With Theorem 2.9 and Theorem 2.5, Lemma 2.5 yields :

• *If  $\alpha$  is not an ordinal  $P_\alpha$  and  $[\omega]^{<\omega}$  are two incomparable minimal members of  $\mathbb{J}_\alpha$ .*

According to Theorem 2.6 and Theorem 2.9:

• *If  $\alpha$  is a countably infinite ordinal,  $Q_\alpha$  and  $P_\alpha$  are minimal obstructions.*

In fact, using this new kind of sierpinskisation, if  $\alpha \leq \omega + \omega = \omega 2$ ,  $Q_\alpha$  and  $P_\alpha$  coincide.

Indeed, if  $\alpha = \omega + n$  with  $n < \omega$ , then  $S_\alpha = \Omega_L(1) \oplus n$ . In this case,  $S_\alpha$  is isomorphic to  $\omega \oplus n$ , hence  $Q_\alpha$  is isomorphic to the direct product  $\omega \times (n+1)$  which in turn is isomorphic to  $\Omega_L(n+1) = P_\alpha$ . If  $\alpha = \omega + \omega = \omega 2$ ,  $S_\alpha = \Omega_L(2)$ . This poset is isomorphic to the direct product  $\omega \times 2$ . In this case,  $Q_\alpha$  is isomorphic to  $[\omega]^2$ , the subset of the product  $\omega \times \omega$  made of pairs  $(i, j)$  with  $i < j$ . In turn this poset is isomorphic to  $\underline{\Omega}_L(\omega) = P_\alpha$ .

Beyond  $\omega 2$ , no  $Q_\alpha$  is a sierpinskisation.

For that, we apply the following refinement of Lemma 2.5 obtained in [9] (see Example 1.2, Chapter 1).

**Proposition 2.1.** *Let  $\gamma$  be a countable order type. Then  $\underline{\Omega}_L(\gamma)$  is embeddable in  $[\omega]^{<\omega}$  as a join-semilattice if and only if  $\gamma \leq \omega$ .*

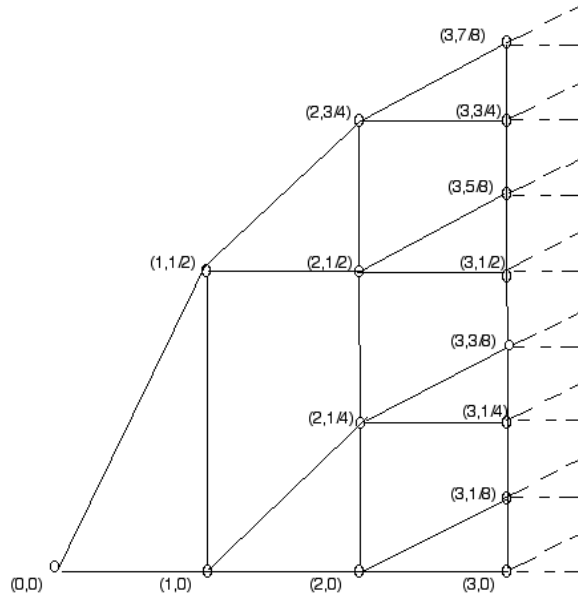
A consequence of Proposition 2.1 is the following:

**Corollary 2.1.** *If  $\alpha$  is a countable chain, then  $P_\alpha$  and  $[\omega]^{<\omega}$  are two incomparable members of  $\mathbb{J}_\alpha$  if and only if  $\alpha$  is not embeddable in  $\omega 2$*

**Proof.** Let  $\alpha \leq \omega 2$ . As we have seen  $P_\alpha$  is isomorphic to  $\omega \times (1 + \alpha')$  if  $\alpha' < \omega$  and to  $[\omega]^2$  if  $\alpha = \omega 2$ . In both cases  $P_\alpha$  is embeddable, as a join-semilattice, in  $[\omega]^{<\omega}$ . Conversely, if  $P_\alpha$  and  $[\omega]^{<\omega}$  are comparable, as join-semilattices, then, necessarily  $P_\alpha$  is embeddable into  $[\omega]^{<\omega}$  as a join-semilattice. From Proposition 2.1, it follows that  $\alpha \leq \omega 2$ .  $\square$

This work leaves open the following questions. We are only able to give some examples of ordinals for which the answer to the first question is positive.

- QUESTIONS 2.3.**
- (1) *If  $\alpha$  is a countably infinite ordinal, does the minimal obstructions are  $\alpha$ ,  $P_\alpha$ ,  $Q_\alpha$  and some lexicographical sums of obstructions corresponding to smaller ordinal?*
  - (2) *If  $\alpha$  is a scattered order type which is not an ordinal, does the minimal obstructions are  $\alpha$ ,  $P_\alpha$ ,  $[\omega]^{<\omega}$  and some lexicographical sums of obstructions corresponding to smaller scattered order types?*
  - (3) *If  $\alpha$  is the order type  $\eta$  of the chain of rational numbers, does  $\mathbb{J}_{\neg\eta} = \text{Forb}_{\mathbb{J}}(\{1 + \eta, [\omega]^{<\omega}, \underline{\Omega}(\eta)\})$  where  $\underline{\Omega}(\eta)$  is the lattice sierpinskisation represented Figure 2.2?*

FIGURE 2.2.  $\underline{\Omega}(\eta)$ 

### 3. Join-semilattices and a proof of Theorem 2.3

A *join-semilattice* is a poset  $P$  such that every two elements  $x, y$  have a least upper-bound, or join, denoted by  $x \vee y$ . If  $P$  has a least element, that we denote 0, this amounts to say that every finite subset of  $P$  has a join. In the sequel, we will mostly consider join-semilattices with a least element. Let  $Q$  and  $P$  be such join-semilattices. A map  $f : Q \rightarrow P$  is *join-preserving* if:

$$(12) \quad f(x \vee y) = f(x) \vee f(y)$$

for all  $x, y \in Q$ .

This map *preserves finite (resp. arbitrary) joins* if

$$(13) \quad f(\bigvee X) = \bigvee \{f(x) : x \in X\}$$

for every finite (resp. arbitrary) subset  $X$  of  $Q$ .

If  $P$  is a join-semilattice with a least element, the set  $J(P)$  of ideals of  $P$  ordered by inclusion is a complete lattice. If  $A$  is a subset of  $P$ , there is a least ideal containing  $A$ , that we denote  $\langle A \rangle$ . An ideal  $I$  is *generated* by a subset  $A$  of  $P$  if  $I = \langle A \rangle$ . If  $[A]^{<\omega}$  denotes the collection of finite subsets of  $A$  we have:

$$(14) \quad \langle A \rangle = \bigvee \{ \bigvee X : X \in [A]^{<\omega} \}$$

**Lemma 2.6.** *Let  $Q$  be a join-semilattice with a least element and  $L$  be a complete lattice. To a map  $g : Q \rightarrow L$  associate  $\bar{g} : \mathfrak{P}(Q) \rightarrow L$  defined by setting  $\bar{g}(X) := \bigvee \{g(x) : x \in X\}$  for every  $X \subseteq Q$ . Then  $\bar{g}$  induces a map from  $J(Q)$  in  $L$  which preserves arbitrary joins whenever  $g$  preserves finite joins.*

**Proof.**

**Claim 2.1.** *Let  $I \in J(Q)$ . If  $g$  preserves finite joins and  $A$  generates  $I$  then  $\bar{g}(I) = \bigvee \{g(x) : x \in A\}$ .*

**Proof of claim 2.1.** Since  $I = \langle A \rangle$  and  $g$  preserves finite joins,  $\langle \{g(x) : x \in I\} \rangle = \langle \{g(x) : x \in A\} \rangle$ . The claimed equality follows.  $\square$

Now, let  $\mathcal{I} \subseteq J(Q)$  and  $I := \bigvee \mathcal{I}$ . Clearly,  $A := \bigcup \mathcal{I}$  generates  $I$ . Claim 2.1 yields  $\bar{g}(I) = \bigvee \{g(x) : x \in A\} = \bigvee \bigcup \{\{g(x) : x \in J\} : J \in \mathcal{I}\} = \bigvee \{\bigvee \{g(x) : x \in J\} : J \in \mathcal{I}\} = \bigvee \{\bar{g}(J) : J \in \mathcal{I}\}$ . This proves that  $\bar{g}$  preserves arbitrary joins.  $\square$

**Remark 2.2.** *Note that in Lemma 2.6 the fact that  $g$  is one-to-one does not necessarily transfer to the map induced by  $\bar{g}$  on  $J(Q)$ . For an example, let  $\kappa := 2^{\aleph_0}$  and let  $Q := [\kappa]^{<\omega}$  and  $L := \mathfrak{P}(\aleph_0)$  ordered by inclusion. The lattice  $J(Q)$  is isomorphic to  $\mathfrak{P}(\kappa)$ , thus  $|Q| = 2^\kappa > \kappa = |L|$ , proving that  $J(Q)$  is not embeddable in  $L$ . On an other hand, as it is well known,  $\mathfrak{P}(\aleph_0)$  contains a subset  $A$  of size  $2^{\aleph_0}$  made of infinite sets which are pairwise almost disjoint. As it is easy to check, the set of finite unions of members of  $A$  is a join-subsemilattice of  $L$  isomorphic to  $Q$ .*

**Lemma 2.7.** *Let  $R$  be a poset and  $P$  be a join-semilattice with a least element. The following properties are equivalent:*

- (i) *There is an embedding from  $I(R)$  in  $J(P)$  which preserves arbitrary joins.*
- (ii) *There is an embedding from  $I(R)$  in  $J(P)$ .*
- (iii) *There is a map  $g$  from  $I_{<\omega}(R)$  in  $P$  such that*

$$(15) \quad X \not\subseteq Y_1 \cup \dots \cup Y_n \Rightarrow g(X) \not\subseteq g(Y_1) \vee \dots \vee g(Y_n)$$

*for all  $X, Y_1, \dots, Y_n \in I_{<\omega}(R)$ .*

- (iv) *There is a map  $h : R \rightarrow P$  such that*

$$(16) \quad \forall i (1 \leq i \leq n \Rightarrow x \not\subseteq y_i) \Rightarrow h(x) \not\subseteq h(y_1) \vee \dots \vee h(y_n)$$

*for all  $x, y_1, \dots, y_n \in R$ .*

**Proof.** (i)  $\Rightarrow$  (ii). Obvious.

(ii)  $\Rightarrow$  (iii). Let  $f$  be an embedding from  $I(R)$  in  $J(P)$ . Let  $X \in I_{<\omega}(R)$ . The set  $A := \text{Max}(X)$  of maximal elements of  $X$  is finite and  $X := \downarrow A$ . Set  $F(X) := \{f(R \setminus \uparrow a) : a \in \text{Max}(X)\}$  and  $C(X) := f(X) \setminus \bigcup F(X)$ .

**Claim 2.2.**  $C(X) \neq \emptyset$  for every  $X \in I_{<\omega}(R)$

**Proof of Claim 2.2.** If  $X = \emptyset$ ,  $F(X) = \emptyset$ . Thus  $C(X) = f(X)$  and our assertion is proved. We may then assume  $X \neq \emptyset$ . Suppose  $C(X) = \emptyset$ , that is  $f(X) \subseteq \bigcup F(X)$ . Since  $f(X)$  is an ideal of  $P$  and  $\bigcup F(X)$  is a finite union of initial segments of  $P$ , this implies that  $f(X)$  is included in some, that is  $f(X) \subseteq f(R \setminus \uparrow a)$  for some  $a \in \text{Max}(X)$ . Since  $f$  is an embedding, this implies  $X \subseteq R \setminus \uparrow a$  hence  $a \notin X$ . A contradiction.  $\square$

Claim 2.2 allows us to pick an element  $g(X) \in C(X)$  for each  $X \in I_{<\omega}(R)$ . Let  $g$  be the map defined by this process. We show that implication (15) holds. Let  $X, Y_1, \dots, Y_n \in I_{<\omega}(R)$ . We have  $g(Y_i) \in f(Y_i)$  for every  $1 \leq i \leq n$ . Since  $f$  is an embedding  $f(Y_i) \subseteq f(Y_1 \cup \dots \cup Y_n)$  for every  $1 \leq i \leq n$ . But  $f(Y_1 \cup \dots \cup Y_n)$  is an ideal. Hence  $g(Y_1) \vee \dots \vee g(Y_n) \in f(Y_1 \cup \dots \cup Y_n)$ . Suppose  $X \not\subseteq Y_1 \cup \dots \cup Y_n$ . There

is  $a \in \text{Max}(X)$  such that  $Y_1 \cup \dots \cup Y_n \subseteq R \setminus \uparrow a$ . And since  $f$  is an embedding,  $f(Y_1 \cup \dots \cup Y_n) \subseteq f(R \setminus \uparrow a)$ . If  $g(X) \leq g(Y_1) \vee \dots \vee g(Y_n)$  then  $g(X) \in f(Y_1 \cup \dots \cup Y_n)$ . Hence  $g(X) \in f(R \setminus \uparrow a)$ , contradicting  $g(X) \in C(X)$ .

(iii)  $\Rightarrow$  (iv). Let  $g : I_{<\omega}(R) \rightarrow P$  such that implication (15) holds. Let  $h$  be the map induced by  $g$  on  $R$  by setting  $h(x) := g(\downarrow x)$  for  $x \in R$ . Let  $x, y_1, \dots, y_n \in R$ . If  $x \not\leq y_i$  for every  $1 \leq i \leq n$ , then  $\downarrow x \not\subseteq (\downarrow y_1 \cup \dots \cup \downarrow y_n)$ . Since  $g$  satisfies implication (15), we have  $h(x) := g(\downarrow x) \not\leq g(\downarrow y_1) \vee \dots \vee g(\downarrow y_n) = h(y_1) \vee \dots \vee h(y_n)$ . Hence implication (16) holds.

(iv)  $\Rightarrow$  (i) Let  $h : R \rightarrow P$  such that implication (16) holds. Define  $f : I(R) \rightarrow J(P)$  by setting  $f(I) := \langle \{h(x) : x \in I\} \rangle$ , the ideal generated by  $\{h(x) : x \in I\}$ , for  $I \in I(R)$ . Since in  $I(R)$  the join is the union,  $f$  preserves arbitrary joins. We claim that  $f$  is one-to-one. Let  $I, J \in I(R)$  such that  $I \not\subseteq J$ . Let  $x \in I \setminus J$ . Clearly  $h(x) \in f(I)$ . We claim that  $h(x) \notin f(J)$ . Indeed, if  $h(x) \in f(J)$ , then  $h(x) \leq \bigvee \{h(y) : y \in F\}$  for some finite subset  $F$  of  $J$ . Since implication (16) holds, we have  $x \leq y$  for some  $y \in F$ . Hence  $x \in J$ , contradiction. Consequently  $f(I) \not\subseteq f(J)$ . Thus  $f$  is one-to-one as claimed.  $\square$

The following proposition rassembles the main properties of the comparizon of join-semilattices.

**Proposition 2.2.** *Let  $P, Q$  be two join-semilattices with a least element. Then:*

- (1)  *$Q$  is embeddable in  $P$  by a join-preserving map iff  $Q$  is embeddable in  $P$  by a map preserving finite joins.*
- (2) *If  $Q$  is embeddable in  $P$  by a join-preserving map then  $J(Q)$  is embeddable in  $J(P)$  by a map preserving arbitrary joins.*
- Suppose  $Q := I_{<\omega}(R)$  for some poset  $R$ . Then:*
- (3)  *$Q$  is embeddable in  $P$  as a poset iff  $Q$  is embeddable in  $P$  by a map preserving finite joins.*
- (4)  *$J(Q)$  is embeddable in  $J(P)$  as a poset iff  $J(Q)$  is embeddable in  $J(P)$  by a map preserving arbitrary joins.*
- (5) *If  $\downarrow x$  is finite for every  $x \in R$  then  $Q$  is embeddable in  $P$  as a poset iff  $J(Q)$  is embeddable in  $J(P)$  as a poset.*

**Proof.**

- (1) Let  $f : Q \rightarrow P$  satisfying  $f(x \vee y) = f(x) \vee f(y)$  for all  $x, y \in Q$ . Set  $g(x) := f(x)$  if  $x \neq 0$  and  $g(0) := 0$ . Then  $g$  preserves finite joins.
- (2) Let  $f : Q \rightarrow P$  and  $\bar{f} : J(Q) \rightarrow J(P)$  defined by  $\bar{f}(I) := \downarrow \{f(x) : x \in I\}$ . If  $f$  preserves finite joins, then  $\bar{f}$  preserves arbitrary joins. Furthermore,  $\bar{f}$  is one to one provided that  $f$  is one-to-one.
- (3) Let  $f : Q \rightarrow P$ . Taking account that  $Q := I_{<\omega}(R)$ , set  $g(\emptyset) := 0$  and  $g(I) := \bigvee \{f(\downarrow x) : x \in I\}$  for each  $I \in I_{<\omega}(R) \setminus \{\emptyset\}$ . Since in  $Q$  the join is the union, the map  $g$  preserves finite unions.
- (4) This is equivalence (i)  $\iff$  (iv) of Lemma 2.7.
- (5) If  $Q$  is embeddable in  $P$  as a poset, then from Item (3),  $Q$  is embeddable in  $P$  by a map preserving finite joins. Hence from Item (1),  $J(Q)$  is embeddable in  $J(P)$  by a map preserving arbitrary joins. Conversely, suppose that  $J(Q)$  is

embeddable in  $J(P)$  as a poset. Since  $J(Q)$  is isomorphic to  $I(R)$ , Lemma 2.7 implies that there is map  $h$  from  $R$  in  $P$  such that implication (16) holds. According to the proof of Lemma 2.7, the map  $f : I(R) \rightarrow J(P)$  defined by setting  $f(I) := \bigvee \{h(x) : x \in I\}$  is an embedding preserving arbitrary joins. Since  $\downarrow x$  is finite for every  $x \in R$ ,  $I$  is finite, hence  $f(I)$  has a largest element, for every  $I \in I_{<\omega}(R)$ . Thus  $f$  induces an embedding from  $Q$  in  $P$  preserving finite joins.

□

**Theorem 2.10.** *Let  $R$  be a poset,  $Q := I_{<\omega}(R)$  and  $\kappa := |Q|$ . Then, there is a set  $\mathbb{B}$ , of size at most  $2^\kappa$ , made of join-semilattices, such that for every join-semilattice  $P$ , the join-semilattice  $J(Q)$  is not embeddable in  $J(P)$  by a map preserving arbitrary joins if and only if no member  $Q$  of  $\mathbb{B}$  is embeddable in  $P$  as a join-semilattice.*

**Proof.** If  $\downarrow x$  is finite for every  $x \in R$  the conclusion of the theorem holds with  $\mathbb{B} = \{Q\}$  (apply Item (3), (4), (5) of Proposition 2.2). So we may assume that  $R$  is infinite. Let  $P$  be a join-semilattice. Suppose that there is an embedding  $f$  from  $J(Q)$  in  $J(P)$  which preserves arbitrary joins.

**Claim 2.3.** *There is a join-semilattice  $Q_f$  such that*

- (1)  $Q_f$  embeds in  $P$  as a join-semilattice.
- (2)  $J(Q)$  embeds in  $J(Q_f)$  by a map preserving arbitrary joins.
- (3)  $|Q_f| = |Q|$ .

**Proof of Claim 2.3.** From Lemma 2.7, there is a map  $g : Q \rightarrow P$  such that inequality (15) holds. Let  $Q_f$  be the join-semilattice of  $P$  generated by  $\{g(x) : x \in Q\}$ . This inequality holds when  $P$  is replaced by  $Q_f$ . Thus from Proposition 2.2,  $J(Q)$  embeds into  $J(Q_f)$  by a map preserving arbitrary joins. Since  $R$  is infinite,  $|Q_f| = |Q| = |R|$ . □

For each join-semilattice  $P$  and each embedding  $f : J(Q) \rightarrow J(P)$  select  $Q_f$ , given by Claim 2.3, on a fixed set of size  $\kappa$ . Let  $\mathbb{B}$  be the collection of this join-semilattices. Since the number of join-semilattices on a set of size  $\kappa$  is at most  $2^\kappa$ ,  $|\mathbb{B}| \leq 2^\kappa$ . □

**Proof of Theorem 2.3.** Let  $\alpha$  be the order type of a chain  $C$ . Apply Theorem above with  $R := C$ . Since, in this case,  $J(Q)$  is embeddable in  $J(P)$  if and only if  $J(Q)$  is embeddable in  $J(P)$  by a map preserving arbitrary joins, Theorem 2.3 follows. □

#### 4. Monotonic sierpinskisations and a proof of Theorem 2.6

In this section, we use the notion of sierpinskisation studied in [34] that we have recalled in the introduction under the name of monotonic sierpinskisation.

We start with some basic properties of ordinary sierpinskisations. For that, let  $\alpha$  be a countably infinite order type. Let  $S$  be a sierpinskisation of  $\alpha$  and  $\omega$ . We assume that  $S = (\mathbb{N}, \leq)$  and the order on  $\mathbb{N}$  is the intersection of a linear order  $\leq_\alpha$  on  $\mathbb{N}$  with the natural order on  $\mathbb{N}$ , such that  $L := (\mathbb{N}, \leq_\alpha)$  has order type  $\alpha$ .

**Lemma 2.8.** *Let  $A$  be a non-empty subset of  $\mathbb{N}$ . Then, the following properties are equivalent:*

- (i) No element of  $A$  is maximal w.r.t.  $S$ .
- (ii) No element of  $A$  is maximal w.r.t.  $L$ .
- (iii)  $A$  is up-directed w.r.t.  $S$  and is infinite.

Furthermore, when one of these conditions holds,  $\downarrow_L A = \downarrow_S A$ .

**Proof.** (i)  $\Rightarrow$  (ii). Observe that, since  $\leq_\alpha$  is a linear extension of  $\leq$ , an element  $a \in A$  which is maximal w.r.t.  $L$  is maximal w.r.t.  $S$ .

(ii)  $\Rightarrow$  (iii). Since  $A$  is non-empty, the inexistence of a maximal element implies that  $A$  is infinite. Let us check that  $A$  is up-directed. Let  $x, y \in A$ . We may suppose  $x \leq_\alpha y$ . Since  $y$  is not maximal in  $A$  w.r.t.  $L$ ,  $\uparrow_L y \cap A$  is infinite. Hence there is some  $z \in \uparrow_L y \cap A$  such that  $x, y \leq_\omega z$ . Clearly,  $x, y \leq z$ , proving that  $A$  is up-directed.

(iii)  $\Rightarrow$  (i) Let  $a \in A$ . If  $a$  is maximal w.r.t.  $S$  then, since  $A$  is up-directed,  $a$  is the largest element of  $A$ . From this fact  $A \subseteq \downarrow_S a$ . Since the natural order on  $\mathbb{N}$  is a linear extension of  $\leq$ ,  $A \subseteq \downarrow_\omega a$ . This latter set being finite,  $A$  is finite, a contradiction.

Let us prove the second assertion. Since  $\leq_\alpha$  is a linear extension of  $\leq$ , we have  $\downarrow_S A \subseteq \downarrow_L A$ . Conversely, let  $x \in \downarrow_L A$ . Let  $y \in A$  such that  $x \leq_L y$ . Since  $A$  satisfies (ii), we proceed as in the proof of (ii)  $\Rightarrow$  (iii). From the fact that  $y$  is not maximal in  $A$  w.r.t.  $L$ ,  $\uparrow_L y \cap A$  is infinite. Hence there is some  $z \in \uparrow_L y \cap A$  such that  $x, y \leq_\omega z$ , proving that  $x \in \downarrow_S A$ .  $\square$

**Proposition 2.3.** *Let  $I$  be a subset of  $\mathbb{N}$ . Then, the following properties are equivalent:*

- (i)  $I$  is a non-principal ideal of  $S$ .
- (ii)  $I$  is an non-empty non-principal initial segment of  $L$ .

Furthermore, when one of these conditions holds, then for every subset  $A$  of  $I$ :

$$I = \downarrow_L A \text{ if and only if } I = \downarrow_S A$$

**Proof.** (i)  $\Rightarrow$  (ii) Assume that  $I$  is a non-principal ideal of  $S$ . According to Lemma 2.8, no element of  $I$  is maximal w.r.t.  $L$ . Moreover  $\downarrow_L I = \downarrow_S I = I$ . Hence,  $I$  is a non-principal initial segment of  $L$ .

(ii)  $\Rightarrow$  (i) Assume that  $I$  is an non-empty non-principal initial segment of  $L$ . Lemma 2.8 yields that  $I$  is up-directed w.r.t.  $S$  and infinite. Since  $\leq_L$  is a linear extension of  $\leq$ ,  $I$  is an initial segment of  $S$ , hence  $I$  is an ideal of  $S$ . From Lemma 2.8 again, it is not principal.

For the second assertion, let  $A$  be a subset of  $I$ . Note that if  $\downarrow_L A = I$ , resp.  $\downarrow_S A = I$ , then no element of  $A$  is maximal w.r.t.  $L$ , resp. w.r.t.  $S$ . Apply Lemma 2.8.  $\square$

**Theorem 2.11.** *Let  $\alpha$  be a countably infinite order type. If  $S$  is a sierpinski-ation of a chain of type  $\alpha$  and a chain of type  $\omega$ , the set  $J^{\downarrow}(S)$  of non-principal ideals of  $S$  forms a chain and this chain has the same order type as the subset of  $I(\alpha)$  made of non-principal initial segments of  $\alpha$ . If  $\alpha = \omega\alpha'$ , every chain  $C \subseteq J(S)$  extends to a chain whose order type is either  $J(\alpha')$  or  $\omega + I(\alpha'')$ , where  $\alpha''$  is a proper final segment of  $\alpha'$ .*

**Proof.** The first sentence is an immediate consequence of the equivalence between (i) and ii) of Proposition 2.3.

Concerning the second sentence, note that  $J^{-\downarrow}(\omega\alpha')$  is isomorphic to  $J(\alpha')$ , hence  $J(S)$  contains a chain of this type. Let  $C \subseteq J(S)$ ,  $C' := \{I \in C : I = \downarrow_S x \text{ for some } x \in S\}$  and  $C'' := \{I \in J^{-\downarrow}(S) : \bigcup C' \subseteq I\}$ . If  $C'$  is empty,  $C''$  contains  $C$  and has order type  $J(\alpha')$ . So in order to complete the proof of the lemma, we may assume that  $C'$  is non-empty.

**Claim 2.4.** *The set  $I_0 := \bigcap C''$  is a non-empty and non-principal ideal of  $S$ .*

**Proof of Claim 2.4.** Clearly  $\bigcup C' \subseteq I_0$ , hence  $I_0$  is non-empty. To see that  $I_0$  is a non-principal ideal, we introduce some notations. We suppose that  $S = (\mathbb{N}, \leq)$  where  $\leq$  is the intersection of the natural order on  $\mathbb{N}$  with a linear order  $\leq_{\omega\alpha'}$  such that  $L := (\mathbb{N}, \leq_{\omega\alpha'})$  has order type  $\omega\alpha'$ , this linear order be given by a bijection  $\varphi$  between  $\mathbb{N}$  and  $\mathbb{N} \times \mathbb{A}'$ . According to Proposition 2.3, each member of  $C''$  is a non-empty non-principal initial segment of  $L$ , thus  $I_0$  is an initial segment of  $L$ . We claim that  $I_0$  is a non-principal initial segment of  $L$ . Suppose for a contradiction that  $I_0 = \downarrow_L x_0$ . Since the order type of  $L$  is  $\omega\alpha'$ , every  $x \in \mathbb{N}$  has a successor  $x'$  w.r.t.  $L$ . Let  $x'_0$  be the successor of  $x_0$ . Then  $x'_0 \notin I$  for some  $I \in C''$ . But since  $I_0 \subseteq I$ , we have  $I_0 = I$ . Thus  $I$  is a principal initial segment of  $L$ . With Proposition 2.3, this contradicts the definition of  $C''$ . Now, since  $I_0$  is a non-empty non-principal initial segment of  $L$ , it follows from Proposition 2.3 that  $I_0$  is a non-principal ideal of  $S$ .  $\square$

**Claim 2.5.** *The chain  $C'$  extends to a chain  $C'_1 \subseteq J(I_0)$  which has order type  $\omega$ . The chain  $C''$  has order type  $I(\alpha'')$ , where  $\alpha''$  is a proper final segment of  $\alpha'$ .*

**Proof of Claim 2.5.** Let  $\nu$  be the order type of  $C'$ . Since each principal initial segment of  $S$  is finite,  $\nu \leq \omega$ . If  $\nu = \omega$ , set  $C'_1 := C'$ . Otherwise, let  $x_0$  be the largest element of  $C'$ . Then  $x_0 \in I_0$ . Apply Claim 2.4. Since  $I_0$  is a non-principal ideal of  $S$ ,  $\uparrow_S x_0 \cap I_0$  contains a chain  $D$  of type  $\omega$ . Set  $C'_1 := C' \cup D$ . Again, since  $I_0$  is a non-principal segment of  $L$ ,  $F := \mathbb{N} \setminus I_0$  is a final segment of  $(\mathbb{N}, \leq_{\omega\alpha'})$  whose image under  $\varphi$  is of the form  $\mathbb{N} \times A''$  where  $A''$  is a proper final segment of  $A'$ . In this case  $C''$  is isomorphic to  $I(A'')$ , hence its order type is  $I(\alpha'')$  where  $\alpha''$  is the order type of  $A''$ .  $\square$

The conclusion of the lemma follows readily from Claim 2.5. Indeed, the set  $C'_1 \cup C''$  is a chain containing  $C$ . According to Claim 2.5, its order type is  $\omega + I(\alpha'')$ .  $\square$

**Corollary 2.2.** *Let  $S$  be a sierpinskiisation of  $\omega\alpha'$  and  $\omega$  then the order types of chains which are embeddable in  $J(S)$  depends only upon  $\alpha'$ . Moreover, if  $\alpha'$  is equimorphic to a chain of order type  $1 + \alpha''$ ,  $\omega + I(\alpha'')$  is the largest order type of the chains of ideals of  $S$ .*

The well-known fact that  $I(\alpha)$  is embeddable in  $I(\beta)$  if and only  $\alpha$  is embeddable in  $\beta$  yields easily the following refinement:

**Lemma 2.9.** *Let  $S$ , resp.  $T$ , be a sierpinskisation of  $\omega\alpha$ , resp.  $\omega\beta$ , and  $\omega$ . Then every chain which is embeddable in  $J(S)$  is embeddable in  $J(T)$  if and only if  $\alpha$  is embeddable in  $\beta$ .*

We recall that for a countable order type  $\alpha'$ , two monotonic sierpinskisations of  $\omega\alpha'$  and  $\omega$  are embeddable in each other and denoted by the same symbol  $\Omega(\alpha')$  and we recall the following result (cf. [34] Proposition 3.4.6. pp. 168)

**Lemma 2.10.** *Let  $\alpha'$  be a countable order type. Then  $\Omega(\alpha')$  is embeddable in every sierpinskisation  $S'$  of  $\omega\alpha'$  and  $\omega$ .*

**Lemma 2.11.** *Let  $\alpha$  be a countably infinite order type and  $S$  be a sierpinskisation of  $\alpha$  and  $\omega$ . Assume that  $\alpha = \omega\alpha' + n$  where  $n < \omega$ . Then there is a subset of  $S$  which is the direct sum  $S' \oplus F$  of a sierpinskisation  $S'$  of  $\omega\alpha'$  and  $\omega$  with an  $n$ -element poset  $F$ .*

**Proof.** Assume that  $S$  is given by a bijective map  $\varphi$  from  $\mathbb{N}$  onto a chain  $C$  having order type  $\alpha$ . Let  $A'$  be the set of the  $n$  last elements of  $C$ ,  $A := \varphi^{-1}(A')$  and  $a$  be the largest element of  $A$  in  $\mathbb{N}$ . The image of  $]a \rightarrow)$  has order type  $\omega\alpha'$ , thus  $S$  induces on  $]a \rightarrow)$  a sierpinskisation  $S'$  of  $\omega\alpha'$  and  $\omega$ . Let  $F$  be the poset induced by  $S$  on  $A$ . Since every element of  $S'$  is incomparable to every element of  $F$  these two posets form a direct sum.  $\square$

**Lemma 2.12.** *Let  $\alpha$  be a countably infinite order type and  $S$  be a sierpinskisation of  $\alpha$  and  $\omega$ . If  $\alpha = \omega\alpha' + n$  where  $n < \omega$  then  $Q_\alpha := I_{<\omega}(\Omega(\alpha') \oplus n)$  is embeddable in  $I_{<\omega}(S)$  by a map preserving finite joins.*

**Proof.**

**Case 1.**  $n = 0$ . By Lemma 2.10  $\Omega(\alpha')$  is embeddable in  $S$ . Thus  $Q_\alpha$  is embeddable in  $I_{<\omega}(S)$  by a map preserving finite joins.

**Case 2.**  $n \neq 0$ . Apply Lemma 2.11. According to Case 1,  $I_{<\omega}(\Omega(\alpha'))$  is embeddable in  $I_{<\omega}(S')$ . On an other hand  $n + 1$  is embeddable in  $I_{<\omega}(F) = I(F)$ . Thus  $Q_\alpha$  which is isomorphic to the product  $I_{<\omega}(\Omega(\alpha')) \times (n + 1)$  is embeddable in the product  $I_{<\omega}(S') \times I_{<\omega}(F)$ . This product is itself isomorphic to  $I_{<\omega}(S' \oplus F)$ . Since  $S' \oplus F$  is embeddable in  $S$ ,  $I_{<\omega}(S' \oplus F)$  is embeddable in  $I_{<\omega}(S)$  by a map preserving finite joins. It follows that  $Q_\alpha$  is embeddable in  $I_{<\omega}(S)$  by a map preserving finite joins.  $\square$

**Theorem 2.12.** *Let  $\alpha$  be a countable ordinal and  $P \in \mathbb{J}_\alpha$ . If  $P$  is embeddable in  $[\omega]^{<\omega}$  by a map preserving finite joins there is sierpinskisation  $S$  of  $\alpha$  and  $\omega$  such that  $I_{<\omega}(S) \in \mathbb{J}_\alpha$  and  $I_{<\omega}(S)$  is embeddable in  $P$  by a map preserving finite joins.*

**Proof.** We construct first  $R$  such that  $I_{<\omega}(R) \in \mathbb{J}_\alpha$  and  $I_{<\omega}(R)$  is embeddable in  $P$  by a map preserving finite joins.

We may suppose that  $P$  is a subset of  $[\omega]^{<\omega}$  closed under finite unions. Thus  $J(P)$  identifies with the set of arbitrary unions of members of  $P$ . Let  $(I_\beta)_{\beta < \alpha+1}$  be a strictly increasing sequence of ideals of  $P$ . For each  $\beta < \alpha$  pick  $x_\beta \in I_{\beta+1} \setminus I_\beta$  and  $F_\beta \in P$  such that  $x_\beta \in F_\beta \subseteq I_{\beta+1}$ . Set  $X := \{x_\beta : \beta < \alpha\}$ ,  $\rho := \{(x_{\beta'}, x_{\beta''}) : \beta' < \beta'' < \alpha \text{ and } x_{\beta'} \in F_{\beta''}\}$ . Let  $\hat{\rho}$  be the reflexive transitive closure of  $\rho$ . Since



$\theta := \{(x_{\beta'}, x_{\beta''}) : \beta' < \beta'' < \alpha\}$  is a linear order containing  $\rho$ ,  $\hat{\rho}$  is an order on  $X$ . Let  $R := (X, \hat{\rho})$  be the resulting poset.

**Claim 2.6.**  $I_{<\omega}(R) \in \mathbb{J}_\alpha$ .

**Proof of claim 2.6.** The linear order  $\theta$  extends the order  $\hat{\rho}$  and has type  $\alpha$ , thus  $I(R)$  has a maximal chain of type  $I(\alpha)$ . Since  $J(I_{<\omega}(R))$  is isomorphic to  $I(R)$ ,  $I_{<\omega}(R)$  belongs to  $\mathbb{J}_\alpha$  as claimed.  $\square$

**Claim 2.7.** For each  $x \in X$ , the initial segment  $\downarrow x$  in  $R$  is finite.

**Proof of claim 2.7.** Suppose not. Let  $\beta$  be minimum such that for  $x := x_\beta$ ,  $\downarrow x$  is infinite. For each  $y \in X$  with  $y < x$  in  $R$  select a finite sequence  $(z_i(y))_{i \leq n_y}$  such that:

- (1)  $z_0(y) = x$  and  $z_{n_y}(y) = y$ .
- (2)  $(z_{i+1}(y), z_i(y)) \in \rho$  for all  $i < n_y$ .

According to item 2,  $z_1(y) \in F_\beta$ . Since  $F_\beta$  is finite, it contains some  $x' := x_{\beta'}$  such that  $z_1(y) = x'$  for infinitely many  $y$ . These elements belong to  $\downarrow x'$ . The fact that  $\beta' < \beta$  contradicts the choice of  $x$ .  $\square$

**Claim 2.8.** Let  $\phi$  be defined by  $\phi(I) := \bigcup \{F_\beta : x_\beta \in I\}$  for each  $I \subseteq X$ . Then:  $\phi$  induces an embedding of  $I(R)$  in  $J(P)$  and an embedding of  $I_{<\omega}(R)$  in  $P$ .

**Proof of claim 2.8.** We prove the first part of the claim. Clearly,  $\phi(I) \in J(P)$  for each  $I \subseteq X$ . And trivially,  $\phi$  preserves arbitrary unions. In particular,  $\phi$  is order preserving. Its remains to show that  $\phi$  is one-to-one. For that, let  $I, J \in I(R)$  such that  $\phi(I) = \phi(J)$ . Suppose  $J \not\subseteq I$ . Let  $x_\beta \in J \setminus I$ . Since  $x_\beta \in J$ ,  $x_\beta \in F_\beta \subseteq \phi(J)$ . Since  $\phi(J) = \phi(I)$ ,  $x_\beta \in \phi(I)$ . Hence  $x_\beta \in F_{\beta'}$  for some  $\beta' \in I$ . If  $\beta' < \beta$  then since  $F_{\beta'} \subseteq I_{\beta'+1} \subseteq I_\beta$  and  $x_\beta \notin I_\beta$ ,  $x_\beta \notin F_{\beta'}$ . A contradiction. On the other hand, if  $\beta < \beta'$  then, since  $x_\beta \in F_{\beta'}$ ,  $(x_\beta, x_{\beta'}) \in \rho$ . Since  $I$  is an initial segment of  $R$ ,  $x_\beta \in I$ . A contradiction too. Consequently  $J \subseteq I$ . Exchanging the roles of  $I$  and  $J$ , yields  $I \subseteq J$ . The equality  $I = J$  follows. For the second part of the claim, it suffices to show that  $\phi(I) \in P$  for every  $I \in I_{<\omega}(R)$ . This fact is a straightforward consequence of Claim 2.7. Indeed, from this claim  $I$  is finite. Hence  $\phi(I)$  is finite and thus belongs to  $P$ .

**Claim 2.9.** The order  $\hat{\rho}$  has a linear extension of type  $\omega$ .

**Proof of claim 2.9.** Clearly,  $[\omega]^{<\omega}$  has a linear extension of type  $\omega$ . Since  $R$  embeds in  $[\omega]^{<\omega}$ , via an embedding in  $P$ , the induced linear extension on  $R$  has order type  $\omega$ .  $\square$

Let  $\rho'$  be the intersection of such a linear extension with the order  $\theta$  and let  $S := (X, \rho')$ .

**Claim 2.10.** For every  $I \in I(S)$ , resp.  $I \in I_{<\omega}(S)$  we have  $I \in I(R)$ , resp.  $I \in I_{<\omega}(R)$ .

**Proof of claim 2.10.** The first part of the proof follows directly from the fact that  $\rho'$  is a linear extension of  $\hat{\rho}$ . The second part follows from the fact that each  $I \in I_{<\omega}(S)$  is finite.  $\square$

It is then easy to check that the poset  $S$  satisfies the properties stated in the theorem.  $\square$

**4.1. Proof of Theorem 2.6.** Apply Theorem 2.12 and Lemma 2.12.  $\square$

### 5. Lattice sierpinskiations and a proof of Theorem 2.9

We show that monotonic sierpinskiations can be identified to special subsets of the direct product of  $\omega$  and  $\alpha$  equipped with the induced ordering. Among these subsets we look at those which are join-subsemilattices of the direct product  $\omega \times \alpha$ , that we call *lattice sierpinskiations*.

**Notation 2.1.** Let  $A$  be a set. Let  $p_1$ , resp.  $p_2$ , be the first, resp. the second projection from the cartesian product  $\mathbb{N} \times A$  onto  $\mathbb{N}$ , resp. onto  $A$ . Let  $S$  be the set of subsets  $S$  of  $\mathbb{N} \times A$  such that:

- (1) Every vertical line  $S(n) := \{\beta \in A : (n, \beta) \in S\}$  is non-empty and finite;
- (2) Every horizontal line  $S^{-1}(a) := \{n \in \mathbb{N} : (n, a) \in S\}$  is an infinite subset of  $\mathbb{N}$ .

We equip  $A$  with a linear order  $\leq$ . Let  $\alpha$  be the order type of  $(A, \leq)$ . The set  $\mathbb{N}$  will be equipped with the natural order, providing a chain of order type  $\omega$ . We equip the cartesian product  $\mathbb{N} \times A$  with the direct product of these two orders, and each subset  $S$  of  $\mathbb{N} \times A$  with the induced order. We denote by  $\mathcal{S}(\alpha)$  the collection of these posets. We denote by  $L_1$  the lexicographic ordering on  $\mathbb{N} \times A$ , that is  $(n', \beta') \leq_{L_1} (n'', \beta'')$  if either  $n' < n''$  w.r.t. the natural ordering on  $\mathbb{N}$  or  $n' = n''$  and  $\beta' \leq \beta''$  w.r.t. the order on  $A$ . And we denote by  $L_2$  the reverse lexicographic ordering on  $\mathbb{N} \times A$ , that is  $(n', \beta') \leq_{L_2} (n'', \beta'')$  if either  $\beta' < \beta''$  w.r.t. the order on  $A$  or  $\beta' = \beta''$  and  $n' \leq n''$  w.r.t. the natural ordering on  $\mathbb{N}$ . With these orders,  $(\mathbb{N} \times A, L_1)$  and  $(\mathbb{N} \times A, L_2)$  have respectively type  $\alpha\omega$  and  $\omega\alpha$ . We consider bijective maps  $\varphi : \mathbb{N} \rightarrow \mathbb{N} \times A$  such that  $\varphi^{-1}$  is order-preserving from  $\mathbb{N} \times \{a\}$  onto  $\mathbb{N}$ , these sets being equipped with the natural ordering.

**Proposition 2.4.** *There is a one-to-one correspondence between members of  $\mathcal{S}(\alpha)$  and monotonic sierpinskiations of  $\omega\alpha$  and  $\alpha$ .*

**Proof.** First, if  $\alpha = 1$ ,  $\mathcal{S}(\alpha) = \{\mathbb{N} \times A\}$ , whereas there is just one monotonic sierpinskiation of  $\alpha$  and  $\omega$ . Hence, we may suppose  $\alpha \neq 1$ . Let  $S \in \mathcal{S}(\alpha)$ . According to item 1 of Notation 2.1,  $(S, L_1|_S)$  has order type  $\omega$ . Let  $\vartheta_1$  be the unique order isomorphism from  $(S, L_1|_S)$  on  $(\mathbb{N}, \leq)$ . Similarly, according to item 2 of Notation 2.1,  $(S, L_2|_S)$  has order type  $\omega\alpha$ . Let  $\vartheta_2$  be the unique order-isomorphism from  $(S, L_2|_S)$  onto  $(\mathbb{N} \times A, L_2)$  such that  $p_2(\vartheta_2(n, \beta)) = \beta$  for all  $\beta \in A$ . And let  $\varphi := \vartheta_2 \circ \vartheta_1^{-1}$ . The chains  $(\mathbb{N}, \leq)$  and  $(\mathbb{N} \times A, L_2)$  have respective order types  $\omega$  and  $\omega\alpha$ . The map  $\varphi$  defines on  $\mathbb{N}$  a monotonic sierpinskiation of  $\omega\alpha$  and  $\omega$ . Clearly,  $\vartheta_1^{-1}$  is an order-isomorphism from  $S$  equipped with the order induced by the direct product  $\omega \times \alpha$  and the monotonic sierpinskiation of  $\omega\alpha$  and  $\alpha$  associated to  $\varphi$ . Conversely, let  $\varphi : \omega \rightarrow \omega\alpha$  be a map defining a monotonic sierpinskiation  $S := (\mathbb{N}, \leq)$ . That is  $\varphi$  is a map from  $(\mathbb{N}, \leq)$  into  $(\mathbb{N} \times A, L_2)$  such that  $\varphi^{-1}$  is order preserving

on each set of the form  $\mathbb{N} \times \{a\}$  for  $a \in A$ . And the order  $\leq$  on  $\mathbb{N}$  is the intersection of the natural order on  $\mathbb{N}$  and the inverse image of the order  $L_2$ .

**Claim 2.11.** *There is a surjective map  $r : \mathbb{N} \rightarrow \mathbb{N}$ , such that for every  $n \in \mathbb{N}$ ,  $r^{-1}(n)$  is the largest initial segment of  $G(n) = \mathbb{N} \setminus \bigcup \{r^{-1}(n') : n' < n\}$  on which  $p_2 \circ \varphi$  is strictly increasing. This map is order preserving.*

**Proof of Claim 2.11.** Applying induction on  $n$ , we may observe that  $\mathbb{N} \setminus G(n)$  is an initial segment of  $\mathbb{N}$  w.r.t. the natural order on  $\mathbb{N}$  and that  $p_2 \circ \varphi$  cannot be strictly increasing on  $G(n)$ .  $\square$

Let  $\theta : \mathbb{N} \rightarrow \mathbb{N} \times A$  defined by setting  $\theta(n) := (r(n), p_2 \circ \varphi(n))$  for every  $n \in \mathbb{N}$ .

**Claim 2.12.**  *$\theta$  is an embedding of  $S$  in  $\mathbb{N} \times A$  equipped with the product ordering. Its image  $S'$  belongs to  $\mathcal{S}(\alpha)$ .*

**Proof of Claim 2.12.** From the fact that  $r^{-1}(n)$  is finite,  $S'(n)$  is finite. Also  $S'^{-1}(a)$  is infinite for every  $a \in A$ . Hence  $S' \in \mathcal{S}(\alpha)$ . The first part of the claim amounts to the fact that  $(n', \varphi(n')) \leq (n'', \varphi(n''))$  is equivalent to  $(r(n'), p_2 \circ \varphi(n')) \leq (r(n''), p_2 \circ \varphi(n''))$ . Suppose  $(n', \varphi(n')) \leq (n'', \varphi(n''))$ . This amounts to  $n' \leq n''$  and  $\varphi(n') \leq \varphi(n'')$ . Since  $r$  and  $p_2$  are order-preserving, we get  $r(n') \leq r(n'')$  and  $p_2 \circ \varphi(n') \leq p_2 \circ \varphi(n'')$ , that is  $(r(n'), p_2 \circ \varphi(n')) \leq (r(n''), p_2 \circ \varphi(n''))$ . Conversely, suppose  $(r(n'), p_2 \circ \varphi(n')) \leq (r(n''), p_2 \circ \varphi(n''))$ . This amounts to  $r(n') \leq r(n'')$  and  $p_2 \circ \varphi(n') \leq p_2 \circ \varphi(n'')$ .

First  $n' \leq n''$ . **Case 1.**  $r(n') = r(n'')$ . Let  $n := r(n')$ . By definition of  $r$ ,  $p_2 \circ \varphi$  is strictly increasing on  $r^{-1}(n)$ . Since  $p_2 \circ \varphi(n') \leq p_2 \circ \varphi(n'')$ , this implies  $n' \leq n''$ . **Case 2.**  $r(n') \neq r(n'')$ . In this case, we have  $r(n') < r(n'')$  and, since  $r$  is order-preserving,  $n' < n''$ .

Next  $\varphi(n') \leq \varphi(n'')$ . **Case 1.**  $p_2 \circ \varphi(n') \neq p_2 \circ \varphi(n'')$ . In this case  $p_2 \circ \varphi(n') < p_2 \circ \varphi(n'')$ . From the definition of the ordering  $\leq_{\omega\alpha}$ ,  $\varphi(n') < \varphi(n'')$ . **Case 2.**  $p_2 \circ \varphi(n') = p_2 \circ \varphi(n'')$ . Since  $\varphi^{-1}$  is order-preserving on each set of the form  $\mathbb{N} \times \{a\}$ , and  $n' \leq n''$ ,  $\varphi(n') \leq \varphi(n'')$ .

From this we get  $(n', \varphi(n')) \leq (n'', \varphi(n''))$  as required.  $\square$

With this claim the proof of the lemma is complete.  $\square$

From now on, we identify monotonic sierpinskisations of  $\omega\alpha$  and  $\omega$  with members of  $\mathcal{S}(\alpha)$ .

**Definition 2.1.** *We say that a member of  $\mathcal{S}(\alpha)$  which is a join-subsemilattice of  $\omega \times \alpha$  is a lattice sierpinskisatation of  $\omega\alpha$  and  $\omega$ . We will denote by  $\mathcal{S}_{Lat}(\alpha)$  the subset of  $\mathcal{S}(\alpha)$  consisting of lattice sierpinskisatations.*

**Lemma 2.13.** *Let  $S$  be a subset of  $\mathbb{N} \times A$  such that:*

- (1) *Every vertical line  $S(n) := \{\beta \in A : (n, \beta) \in S\}$  is non-empty and finite;*
- (2) *Every horizontal line  $S^{-1}(a) := \{n \in \mathbb{N} : (n, a) \in S\}$  is a cofinite subset of  $\mathbb{N}$ .*

*Then,  $S$  equipped with the order induced by the direct product  $\omega \times \alpha$  belongs to  $\mathcal{S}_{Lat}(\alpha)$ .*

**Proof.** The fact that  $S$  is a join-subsemilattice of  $\mathbb{N} \times A$  follows from the second condition. Indeed, let  $x := (n, \beta), y := (m, \gamma) \in S$ . W.l.o.g. we may assume  $\beta \leq \gamma$ .

If  $n \leq m$  we have  $x \leq y$  and their supremum is  $y$ . If  $m \not\leq n$  then  $n < m$ . According to Condition (2),  $z := (n, \beta) \in S$ . Since  $z$  is the supremum of  $x$  and  $y$  in  $\omega \times \alpha$ ,  $z$  is their supremum in  $S$ .  $\square$

**Proposition 2.5.** *If  $S, S' \in \mathcal{S}_{Lat}(\alpha)$  there is a map  $t : \mathbb{N} \rightarrow \mathbb{N}$  preserving the natural order such that the map  $(t, 1_A)$  induces a join-embedding map from  $S$  to  $S'$*

**Proof.**

**Claim 2.13.** *For every  $n, n' \in \mathbb{N}$  such that  $n < n'$  there is some  $n'' \in \mathbb{N}$  such that  $n'' > n'$  and  $S(n) \subseteq S'(n'')$ .*

**Proof of Claim 2.13.** Since  $S'^{-1}(a)$  is infinite for every  $a \in A$ , we may select  $n'_a > n'$  such that  $(n'_a, a) \in S'$  for every  $a \in A$ . Let  $m := \text{Max}\{n'_a : a \in S(n)\}$ . Let  $a_0$  be the least element of  $S(n)$  w.r.t. the ordering on  $A$ . Let  $n'' \geq m$  such that  $(n'', a_0) \in S'$ . Then  $S(n) \subseteq S'(n'')$ . Indeed, let  $a \in S(n)$ . If  $a = a_0$ ,  $a \in S'(n'')$  by definition. If  $a \neq a_0$  then, since  $(n_a, a) \in S'$ ,  $(n'', a_0) \vee (n_a, a) = (n'', a) \in S'$ , hence  $a \in S'$  as required.  $\square$

Claim 2.13 allows us to define  $t$  inductively. We suppose  $t$  defined for all  $m \in \mathbb{N}$  such that  $m < n$ . From Claim 2.13 there is some  $n''$  such that  $n'' > t(m)$  for all  $m < n$  and  $S(n) \subseteq S'(n'')$ . We set  $t(n) := n''$  where  $n''$  is the least  $n''$  satisfying this property. The pair  $(t, 1_A)$  is a join-embedding map from  $S$  to  $S'$ . Indeed, let  $(n, a), (n', a') \in S$ . W.l.o.g. we may assume  $n \leq n'$ , hence  $(n, a) \vee (n', a') = (n', a'')$ , where  $a'' := \text{Max}_A(\{a, a'\})$ . Since  $S(n') \subseteq S'(t(n'))$ ,  $a'' \in S'(t(n'))$ , hence  $(t(n), a) \vee (t(n'), a') = (t(n'), a'')$ , as required.  $\square$

**Notation 2.2.** *Since all members of  $\mathcal{S}_{Lat}$  are embeddable in each others as join-semilattices, we may denote by a single expression, namely  $\Omega_L(\alpha)$ , an arbitrary member  $S$  of  $\mathcal{S}_{Lat}(\alpha)$  and by  $\underline{\Omega}_L(\alpha)$  the join-semilattice  $\underline{S}$  obtained by adding a least element to  $S$  (if it does not have one). Since for each  $x \in \underline{S}$  the initial segment  $\downarrow x$  is finite,  $\underline{S}$  is in fact a lattice (but not necessarily a sublattice of  $\omega \times \alpha$ ) hence the name of lattice sierpinskiisation.*

**Example 2.1.** *If  $\alpha = 1$ , monotonic and lattice sierpinskiisations of  $\omega$  and  $\omega$  are isomorphic to  $\omega$ , thus we have  $\Omega_L(1) = \Omega(1) = \omega$ . If  $\alpha = n$ , with  $0 < n < \omega$ , the direct product  $\omega \times n$  is a lattice sierpinskiisation of  $\omega n$  and  $\omega$ . There are others, but we will not hesitate to write  $\Omega_L(n) = \omega \times n$ . If  $\alpha = \omega$ , the subset  $X := \{(i, j) : j \leq i < \omega\}$  of the direct product  $\omega \times \omega$  is a lattice sierpinskiisation of  $\omega^2$  and  $\omega$ . This is a lattice isomorphic to  $[\omega]^2$ , the set of pairs  $(i, j)$  such that  $i < j < \omega$  componentwise ordered, and we will write  $\Omega_L(\omega) = [\omega]^2$ . We may also observe that  $S(\omega^*)$  contains the join-semilattice  $\Omega(\omega^*)$  defined previously. Also,  $S(1 + \eta)$  contains the lattice represented Figure 2.2 and denoted by  $\Omega(\eta)$  as well as  $\underline{\Omega}(\eta)$ .*

**Proposition 2.6.** *Let  $(A, \leq)$  be a chain and  $S$  be a join-subsemilattice of the product  $\mathbb{N} \times A$  equipped with the product ordering. Let  $A' := p_2(S)$ ,  $F := \{a' \in A' : S^{-1}(a') \text{ is infinite}\}$ ,  $I := A' \setminus F$  and  $S_I := S \cap (\mathbb{N} \times I)$ . Let  $\alpha$  be the order type of  $(F, \leq_{\uparrow F})$  and  $m$  be the length of the longest chain in  $S_I$  if  $S_I$  is finite. If the vertical lines  $S(n)$  of  $S$  are finite for all integers  $n$  then  $S$  contains a join-subsemilattice  $S'$  such that*

- (1)  $S'$  is isomorphic to  $m + S''$  where  $S'' \in \Omega_L(\alpha)$  if  $S_I$  is finite.
- (2)  $S' \in \Omega_L(1 + \alpha)$  if  $S_I$  is infinite.

Moreover, a chain is embeddable in  $J(S)$  if and only if it is embeddable in  $J(S')$ .

**Proof.**

**Claim 2.14.** *Let  $a \in A$  such that  $S^{-1}(a) \neq \emptyset$ . Then  $(\mathbb{N} \times \{a\}) \cap S$  is cofinal in  $(\mathbb{N} \times (\leftarrow a]) \cap S$ .*

**Proof of Claim 2.14.** Let  $(n', a') \in (\mathbb{N} \times (\leftarrow a]) \cap S$ . Let  $n \in \mathbb{N}$  such that  $(n, a) \in S$ . Since  $S$  is a join-subsemilattice of  $\mathbb{N} \times A$ ,  $(n', a') \vee (n, a) = (\text{Max}\{n, n'\}, a) \in S$ . Hence,  $(n', a')$  is majorized by  $(\text{Max}\{n, n'\}, a) \in (\mathbb{N} \times \{a\}) \cap S$ . This proves that  $(\mathbb{N} \times \{a\}) \cap S$  is cofinal.  $\square$

**Claim 2.15.** *The set  $F$  is a final segment of  $A'$  equipped with the order induced by the order on  $A$ .*

**Proof of Claim 2.15.** Let  $a' \in F$  and  $a \in A'$  such that  $a' \leq a$ . Clearly, each element  $(a, n) \in S$  dominates only finitely many elements  $(a', n') \in S$ . According to Claim 2.14,  $(\mathbb{N} \times \{a\}) \cap S$  is cofinal, thus if  $S^{-1}(a)$  is finite,  $S^{-1}(a')$  is finite too. A contradiction. Hence,  $a \in F$ . Proving that  $F$  is a final segment.  $\square$

Let  $S_F := S \cap (N \times F)$ .

**Claim 2.16.**  *$S_I$  is a join-subsemilattice of  $S$  and if  $N'$  is a final segment of  $p_1(S_F)$  equipped with the natural order on  $\mathbb{N}$ ,  $S \cap (N' \times F)$  is a join-subsemilattice of  $S$ . Furthermore, if the vertical lines  $S(n)$  are finite for all integers  $n$ ,  $S \cap (N' \times F)$  is a lattice sierpinskisation of  $\omega\alpha$  and  $\omega$ .*

**Proof of Claim 2.16.** Since  $F$  is a final segment of  $A'$  (Claim 2.15),  $I$  is an initial segment of  $A'$ , hence  $S_I$  is a join-subsemilattice of  $S$ . By the same token  $S \cap (N' \times F)$  is a join-subsemilattice of  $S$ . If the vertical lines  $S(n)$  are finite for all integers  $n$ , then  $S \cap (N' \times F)$  satisfies the conditions of Notation 2.1 with  $N'$  and  $F$  instead of  $\mathbb{N}$  and  $A$ .  $\square$

From now on, we suppose that the vertical lines  $S(n)$  are finite for all integers  $n$ .

**Claim 2.17.** *Every proper ideal of  $S_I$  is finite.*

**Proof of Claim 2.17.** We prove first that if  $\nu$  is the order type of  $(I, \leq_I)$  then  $\nu \leq \omega$ . For that, it suffices to prove that for every  $a \in I$ , the initial segment  $(\leftarrow a] \cap A'$  is finite. Let  $a \in I$ . With the fact that all the vertical lines  $S(n)$  are finite, we get that the initial segment of  $\mathbb{N} \times A$  (equipped with the product order) generated by  $(\mathbb{N} \times \{a\}) \cap S$  is finite. Since from Claim 2.14,  $(\mathbb{N} \times \{a\}) \cap S$  is cofinal in  $(\mathbb{N} \times (\leftarrow a]) \cap S$ , it follows that this latter set is finite. Thus, its second projection is finite too. This second projection being  $(\leftarrow a] \cap A'$ , our assertion is proved. Now, let  $I'$  be a proper ideal of  $S_I$ . Suppose by contradiction that  $I'$  is infinite. Then necessarily,  $p_2(I') \neq I$ . Otherwise since  $I'$  is a proper ideal of  $S_I$ ,  $p_1(I') \neq p_1(S_I)$ . Since  $p_1(I')$  is an initial segment of  $p_1(S_I)$ ,  $p_1(I')$  is finite; since each vertical line  $S(n)$  for  $n \in p_1(I')$  is finite,  $I'$  is finite. Now, pick  $a \in I \setminus p_2(I')$ . As above,

$(\mathbb{N} \times \{a\}) \cap S$  is cofinal in  $(\mathbb{N} \times (\leftarrow a]) \cap S$ , thus this set, and in particular  $I'$  is finite.

□

With these claims, the proof of the proposition goes as follows:

**Case 1.**  $S_I$  is finite. In this case, let  $N' := p_1(S_F) \cap [n \rightarrow)$  where  $n = 0$  if  $S_I$  is empty and  $n = p_1(x)$  where  $x$  is the largest element of  $S_I$  otherwise. Let  $M$  be the largest sized subchain of  $S_I$ ,  $m := |M|$ , let  $S'' := S \cap (N' \times F)$  and let  $S' := M + S''$ .

**Case 2.**  $S_I$  is infinite. In this case, it follows from Claim 2.17 that  $S_I$  contains a cofinal chain of type  $\omega$ . Let  $D$  be such a chain. Add an extra element  $\{a\}$  to  $F$  with the requirement that  $a \leq b$  for all  $b \in F$ , set  $N' := p_1(S_F)$ ,  $S'' := S_F$ . Let  $S'$  be the subset of  $\mathbb{N} \times (\{a\} \cup F)$  made of  $S''$  and  $(p_1(D) \cap N') \times \{a\}$ .

In case 1, it follows from Claim 2.16 that  $S''$  is a lattice sierpinskiisation of  $\omega\alpha$  and  $\omega$ . By construction  $S'$  is a join-subsemilattice of  $S$  isomorphic to  $m + S''$ . In case 2,  $S'$  is a lattice sierpinskiisation of  $\omega(1 + \alpha)$  and  $\omega$ . And one can check that  $S'$  is embeddable in  $S$  as join-semilattice.

To conclude, we only need to check that the same chains are embeddable in  $J(S)$  and  $J(S')$ . Let  $C \subseteq J(S)$  be a chain. Set  $C'' := C \cap (\mathbb{N} \times F)$  and  $C' := C \cap (\mathbb{N} \times I)$ . Since  $S \cap (N' \times F)$  and  $S''$  are sierpinskiisations of  $\omega\alpha$  and  $\omega$  (Claim 2.16), it follows from Corollary 2.2 that  $C''$  is embeddable in  $J(S'')$ . In case 1, since  $|C'| \leq |M|$ , it follows that  $C = C' + C''$  is embeddable in  $J(S)$ . In case 2, the order type of  $C'$  is at most  $\omega + 1$ . By the same token,  $C$  is embeddable in  $J(S')$ . □

□

**Proof of Theorem 2.9.** Let  $\alpha$ . First,  $P_\alpha \in \mathbb{J}_\alpha$ . This follows readily from the fact that  $n + \underline{\Omega}_L(\alpha') \in \mathbb{J}_{n+\alpha'}$ ,  $\underline{\Omega}_L(1 + \alpha') \in \mathbb{J}_{\omega+\alpha'}$  and  $\underline{\Omega}_L(\alpha') \in \mathbb{J}_{\alpha'}$  for every  $\alpha'$  and  $n < \omega$ . Next, let  $S$  be a join-subsemilattice of  $P_\alpha$  having a least element and belonging to  $\mathbb{J}_\alpha$ . Clearly, the join-semilattice  $P_\alpha$  is embeddable as a join-semilattice in a product  $A \times \mathbb{N}$ , where  $A$  is a chain. Let  $S'$  be a join-subsemilattice of  $S$  satisfying the properties of Proposition 2.6. Clearly  $J(S') \in \mathbb{J}_\alpha$ . We may suppose that  $S'$  has a least element (otherwise, add the least element of  $S$ ). Thus the join-semilattice  $S'$  is of the form  $m + \underline{\Omega}_L(\beta)$  with  $m < \omega$ . We claim that  $P_\alpha$  is embeddable in  $S'$  by a join-preserving map. If  $\alpha + 1 \leq \alpha$ ,  $P_\alpha$  is of the form  $\underline{\Omega}_L(\gamma)$ . Lemma 2.9 yields that  $\gamma$  and  $\beta$  are equimorphic. If  $\alpha + 1 \not\leq \alpha$ ,  $P_\alpha := n + \underline{\Omega}_L(\alpha')$  where  $n < \omega$ ,  $\alpha'$  has no least element and  $\alpha = n + \alpha'$ . In this case, using Lemma 2.9 one obtains that  $m \geq n$  and  $\beta$  is equimorphic to  $\alpha$ . In both cases the existence of an embedding follows. □

## 6. Some examples of obstructions

As already observed, we have  $J_{-\alpha} = \text{Forb}_{\mathbb{J}}(\{1 + \alpha\})$  for every  $\alpha \leq \omega$ . The first interesting case is  $\alpha = \omega + 1$ . We have:

**Lemma 2.14.**

$$(17) \quad \mathbb{J}_{-(\omega+1)} = \text{Forb}_{\mathbb{J}}(\{\omega + 1, \omega \times 2\}) = \text{Forb}_{\mathbb{J}}(\{\omega + 1, Q_{\omega+1}\}).$$

**Proof.** Let  $P \in \mathbb{J}_\alpha$  such that  $\omega + 1 \not\leq P$ . In  $J(P)$  we have a strictly increasing chain  $(I_\gamma)_{\gamma < \omega+2}$ . We construct in  $P$  a join-subsemilattice  $\{x_{i,j} : 0 \leq i \leq 1, j < \omega\}$  isomorphic to  $\omega \times 2$  such that  $x_{1,j} \in (I_{\omega+1} \setminus I_\omega) \cap \uparrow x_{0,j}$  for every  $j < \omega$ . Pick  $x_{0,0}$  in  $I_0$  and  $x_{1,0}$  in  $(I_{\omega+1} \setminus I_\omega) \cap \uparrow x_{0,0}$ . Suppose  $x_{0,0}, x_{1,0}, \dots, x_{0,j}, x_{1,j}$  constructed.

Since  $\omega + 1 \not\leq P$  and  $x_{1,j} \in (I_{\omega+1} \setminus I_\omega) \cap \uparrow x_{0,j}$ , there is  $n_j < \omega$  such that  $x_{1,j}$  does not dominate  $I_n \cap \uparrow x_{0,j}$  for  $n \geq n_j$ . Pick  $x_{0,j+1} \in I_{n_j} \cap \uparrow x_{0,j}$  and put  $x_{1,j+1} := x_{0,j+1} \vee x_{1,j}$ .

□

Next, we show:

$$(18) \quad \mathbb{J}_{-(\omega+2)} = \text{Forb}_{\mathbb{J}}(\{\omega + 2, (\omega \times 2) + 1, \omega \times 3\})$$

More generally, we solve the case  $\omega + n$ .

**Lemma 2.15.** *Let  $n < \omega$  and  $1 \leq k \leq n + 1$ . Then  $\Omega(k) + n + 1 - k \in \mathbb{J}_{\omega+n}$ .*

**Proof.** We may suppose  $\Omega(k) = \omega \times k$ . Hence  $J(\Omega(k) + n + 1 - k) = J((\omega \times k) + n + 1 - k) = (J(\omega) \times J(k)) + n + 1 - k = ((\omega + 1) \times k) + n + 1 - k$ . Hence  $I(\omega + n) = \omega + n + 1 \leq J((\omega \times k) + n + 1 - k)$ . □

**Lemma 2.16.** *Let  $\alpha := \omega + n$  with  $n \geq 1$ . Then:*

$$(19) \quad \mathbb{J}_{-\alpha} = \text{Forb}_{\mathbb{J}}(\{\Omega(k) + n + 1 - k : 1 \leq k \leq n + 1\}).$$

**Proof.** Lemma 2.15 implies  $\Omega(k) + n + 1 - k \in \mathbb{J}_{\omega+n}$  for every  $1 \leq k \leq n + 1$ . To prove that this is a complete set of obstructions, we proceed by recurrence on  $n$ . The case  $n = 1$  is solved in Lemma 2.14. Suppose  $\mathbb{J}_{\omega+n} = \uparrow \{\Omega(k) + n + 1 - k : 1 \leq k \leq n + 1\}$ . Let  $P \in \mathbb{J}_{\omega+n+1}$  such that  $P$  contains no join-subsemilattice isomorphic to  $\Omega(k) + n + 2 - k$  for every  $1 \leq k \leq n + 1$ . In  $J(P)$  we have a strictly increasing chain  $(I_\gamma)_{\gamma < \omega+n+2}$ . Pick  $x \in I_0$  and  $x' \in (I_{\omega+n+1} \setminus I_{\omega+n}) \cap \uparrow x$ . Put  $P' := [x, x']$ . Clearly  $P' \in \mathbb{J}_{\omega+n}$ . Since  $P$  does not contain a join-subsemilattice isomorphic to  $\Omega(k) + n + 2 - k$  for every  $1 \leq k \leq n + 1$ ,  $P'$  does not contain a join-subsemilattice isomorphic to  $\Omega(k) + n + 1 - k$ , for every  $1 \leq k \leq n$ . Recurrence's hypothesis implies that  $P'$  contains a join-subsemilattice  $\{z_{i,j} : 0 \leq i \leq n, j < \omega\}$  isomorphic to  $\Omega(n + 1)$ . We construct in  $P$  a join-subsemilattice  $\{x_{i,j} : 0 \leq i \leq n + 1, j < \omega\}$  isomorphic to  $\Omega(n + 2)$  such that a) for  $0 \leq i \leq n$ ,  $x_{i,j} = z_{i,k}$  for some  $k \geq j$  and b)  $x_{n+1,j} \in (I_{\omega+n+1} \setminus I_{\omega+n}) \cap \uparrow x_{n,j}$ . Put  $x_{i,0} := z_{i,0}$  for  $0 \leq i \leq n$  and pick  $x_{n+1,0} \in (I_{\omega+n+1} \setminus I_{\omega+n}) \cap \uparrow x_{n,0}$ . Suppose  $x_{i,j}$  constructed for  $0 \leq i \leq n + 1$  and  $0 \leq j \leq m$ . Since  $P$  does not contain a join-subsemilattice isomorphic to  $\Omega(n + 1) + 1$  and  $x_{n+1,m} \in (I_{\omega+n+1} \setminus I_{\omega+n}) \cap \uparrow x_{n,m}$  there is  $j_m > m$  such that  $x_{n+1,m} \not\leq z_{n,j}$  for  $j \geq j_m$ . Put  $x_{i,m+1} := z_{i,j_m}$  for  $0 \leq i \leq n$  and  $x_{n+1,m+1} := x_{n,m+1} \vee x_{n+1,m}$ . □

**Lemma 2.17.**  $\mathbb{J}_{-\omega 2} = \text{Forb}_{\mathbb{J}}(\{\omega 2, \underline{\Omega}(\omega)\})$ .

**Proof.** Let  $P \in \mathbb{J}_{\omega 2}$ . Suppose  $\omega 2 \not\leq P$ . Let  $(I_\gamma)_{\gamma < \omega 2}$  be a strictly increasing chain in  $J(P)$ . We construct a join-subsemilattice  $\{x_{i,j} : i \leq j < \omega\}$  of  $P$ , isomorphic to  $\underline{\Omega}(\omega)$  such that  $x_{i,i} \in (I_{\omega+i} \setminus I_{\omega+i-1}) \cap \uparrow x_{i-1,i}$  for every  $i < \omega$ . Pick  $x_{0,0} \in I_0$ ,  $x_{0,1} \in I_1 \cap \uparrow x_{0,0}$  and  $x_{1,1} \in (I_{\omega+1} \setminus I_\omega) \cap \uparrow x_{0,1}$ . Suppose  $x_{i,j}$  constructed for  $0 \leq i \leq j \leq n$ . Since  $\omega 2 \not\leq P$  and  $x_{n,n} \in (I_{\omega+n} \setminus I_{\omega+n-1}) \cap \uparrow x_{n-1,n}$ , we have  $\omega + 1 \not\leq x_{n,n}$ . Hence, there is some  $j_n < \omega$  such that  $x_{n,n}$  does not dominate  $I_j \cap \uparrow x_{0,n}$  for all  $j$  such that  $j_n \leq j < \omega$ . Pick  $x_{0,n+1} \in I_{j_n} \cap \uparrow x_{0,n}$ , put  $x_{i+1,n+1} := x_{i,n+1} \vee x_{i+1,n}$  for  $0 \leq i \leq n - 1$  and pick  $x_{n+1,n+1} \in (I_{\omega+n+1} \setminus I_{\omega+n}) \cap \uparrow x_{n,n+1}$ . □





## CHAPTER 3

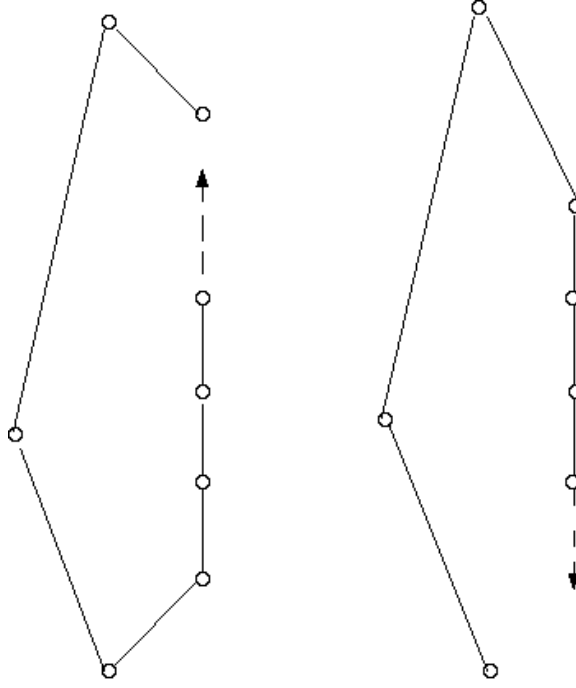
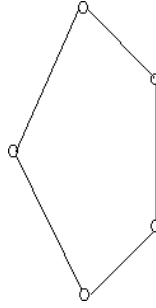
### The length of chains in modular algebraic lattices

We show that, for a large class of countable order types  $\alpha$ , a modular algebraic lattice  $L$  contains no chain of type  $\alpha$  if and only if  $K(L)$ , the join-semilattice of compact elements of  $L$ , contains neither a chain of type  $\alpha$  nor a subset isomorphic to  $[\omega]^{<\omega}$ , the set of finite subsets of  $\omega$ .

#### 1. Introduction and presentation of the results

This paper is about the relationship between the order structure of an algebraic lattice  $L$  and the order structure of the join-semilattice  $K(L)$ , made of the compact elements of  $L$ , particularly, the relationship between the length of chains in  $L$  and in  $K(L)$ . The lattice  $L$  is isomorphic to  $J(K(L))$ , the collection of ideals of  $K(L)$  ordered by inclusion, but this does not make this relationship immediately apparent. For an example, if  $L$  is  $\mathfrak{P}(E)$ , the power set of a set  $E$ , then  $K(E)$  is  $[E]^{<\omega}$ , the collection of finite subsets of  $E$ . If  $E$  is infinite, chains in  $[E]^{<\omega}$  and in  $\mathfrak{P}(E)$  are quite far apart: maximal chains in  $[E]^{<\omega}$  have order type  $\omega$  whereas in  $\mathfrak{P}(E)$  each maximal chain is made of the set  $I(C)$  of initial segments of a chain  $C := (E, \leq)$  where  $\leq$  is a linear order on  $E$ . And, from this follows that  $\mathfrak{P}(E)$  contains uncountable chains. On an other hand, there are classes of lattices  $L$  for which, except this case, chains in  $L$  and in  $K(L)$  are about the same. Our results are about a class  $\mathbb{L}$  of algebraic lattices, including the modular ones.

Let  $\alpha$  be a chain, we denote by  $I(\alpha)$  the set of initial segments of  $\alpha$ , ordered by inclusion, we denote by  $L_\alpha := 1 + (1 \oplus \alpha) + 1$  the lattice made of the direct sum of the one-element chain 1 and the chain  $\alpha$ , with top and bottom added. Note that

FIGURE 3.1.  $L_{\omega+1}, L_{\omega^*}$ FIGURE 3.2.  $M_5$ 

$I(\alpha) \cong J(1 + \alpha)$  and that the algebraic lattice  $J(L_\alpha)$  made of the set of ideals of  $L_\alpha$ , ordered by inclusion, is isomorphic to  $L_{J(\alpha)}$ .

Let  $\mathbb{A}$  be the class of algebraic lattices and let  $\mathbb{L}$  be the collection of  $L \in \mathbb{A}$  such that  $L$  contains no sublattice isomorphic to  $L_{\omega+1}$  or to  $L_{\omega^*}$ . Since  $L_2$  is isomorphic to  $M_5$ , the five element non-modular lattice, every modular algebraic lattice belongs to  $\mathbb{L}$ . Let  $\mathbb{A}_{-\alpha}$  (resp.  $\mathbb{L}_{-\alpha}$ ) be the collection of  $L \in \mathbb{A}$  (resp.  $L \in \mathbb{L}$ ) such that  $L$  contains no chain of type  $I(\alpha)$ . Let  $\mathbb{K}$  be the class of order types  $\alpha$  such that  $L \in \mathbb{L}_{-\alpha}$  whenever  $K(L)$  contains no chain of type  $1 + \alpha$  and no subset isomorphic to  $[\omega]^{<\omega}$ . If  $\alpha$  is countable then these two conditions are necessary in order that  $L \in \mathbb{L}_{-\alpha}$ ; if, moreover,  $\alpha$  is indecomposable, this is equivalent to the fact that  $L$  contains no chain of type  $\alpha$ .

**Theorem 3.1.** *The class  $\mathbb{K}$  satisfies the following properties:*

- (p<sub>1</sub>)  $0 \in \mathbb{K}$ ,  $1 \in \mathbb{K}$ ;
- (p<sub>2</sub>) *If  $\alpha + 1 \in \mathbb{K}$  and  $\beta \in \mathbb{K}$  then  $\alpha + 1 + \beta \in \mathbb{K}$ ;*
- (p<sub>3</sub>) *If  $\alpha_n + 1 \in \mathbb{K}$  for every  $n < \omega$  then the  $\omega$ -sum  $\gamma := \alpha_0 + 1 + \alpha_1 + 1 + \dots + \alpha_n + 1 + \dots$  belongs to  $\mathbb{K}$ ;*
- (p<sub>4</sub>) *If  $\alpha_n \in \mathbb{K}$  and  $\{m : \alpha_n \leq \alpha_m\}$  is infinite for every  $n < \omega$  then the  $\omega^*$ -sum  $\delta := \dots + \alpha_{n+1} + \alpha_n + \dots + \alpha_1 + \alpha_0$  belongs to  $\mathbb{K}$ .*
- (p<sub>5</sub>) *If  $\alpha$  is a countable scattered order type, then  $\alpha \in \mathbb{K}$  if and only if  $\alpha = \alpha_0 + \alpha_1 + \dots + \alpha_n$  with  $\alpha_i \in \mathbb{K}$  for  $i \leq n$ ,  $\alpha_i$  strictly left-indecomposable for  $i < n$  and  $\alpha_n$  indecomposable;*
- (p<sub>6</sub>)  $\eta \in \mathbb{K}$ .

**Corollary 3.1.** *A modular algebraic lattice is well-founded, respectively scattered, if and only if the join-semilattice made of its compact element is well-founded, resp. scattered and does not contains a join-subsemilattice isomorphic to  $[\omega]^{<\omega}$ .*

Let  $\mathbb{P}$  be the smallest class of order-types satisfying properties (p<sub>1</sub>) to (p<sub>4</sub>) above. For example,  $\omega, \omega^*, \omega(\omega^\alpha)^*, \omega^*\omega, \omega^*\omega\omega^*$  belong to  $\mathbb{P}$ . We have  $\mathbb{P} \subseteq \mathbb{K} \setminus \{\eta\}$ . We do not know whether every countable  $\alpha \in \mathbb{K} \setminus \{\eta\}$  is equimorphic to some  $\alpha' \in \mathbb{P}$ . As a matter of fact, the class of order types  $\alpha$  which are equimorphic to some  $\alpha' \in \mathbb{P}$  satisfies also (p<sub>5</sub>) hence, to answer our question, we may suppose that  $\alpha$  is indecomposable. We only succeed when  $\alpha$  is indivisible. We eliminate those outside  $\mathbb{P}$  by building algebraic lattices of the form  $J(T)$  where  $T$  is an appropriate distributive lattices.

**Theorem 3.2.** *A countable indivisible order type  $\alpha$  is equimorphic to some  $\alpha' \in \mathbb{P} \cup \{\eta\}$  if and only if for every modular algebraic lattice  $L$ ,  $L$  contains a chain of type  $I(\alpha)$  if and only if  $L$  contains a chain of type  $1 + \alpha$  or a subset isomorphic to  $[\omega]^{<\omega}$ .*

This contains the fact that for a countable ordinal  $\alpha$ ,  $\alpha \in \mathbb{K}$  if and only if  $\alpha \leq \omega$ , a fact which can be obtained directly by a straightforward sierpinskiization.

A characterization, by means of obstructions, of posets with no chain of ideals of a given type is given in [34]. The motivation for our results can be found in [2], a paper describing a characterization of countable order types  $\alpha$  such that the lattice  $I(P)$  of initial segments of a poset  $P$  contains no chain of order type  $I(\alpha)$  if and only if  $P$  contains no chain of order type  $\alpha$  and no infinite antichain (see [4] for a proof). Results presented here were obtained first for the special case of distributive algebraic lattices. They were included in the thesis of the first author presented before the University Claude-Bernard in december 1992[6], and announced in [7].

## 2. Definitions and basic notions

Our definitions and notations are standard and agree with [15] and [38] except on minor points that we will mention.

**2.1. Posets, well-founded and scattered posets, ordinals and scattered order types.** Let  $P$  be a set. A *quasi-order* on  $P$  is a binary relation  $\rho$  on  $P$  which is reflexive and transitive ; the set  $P$  equipped with this quasi-order is a *qoset*. As usual,  $x \leq y$  stands for  $(x, y) \in \rho$ . An antisymmetric quasi-order is an *order* and  $P$  equipped with this order is a *poset*. The *dual* of a poset  $P$ , denoted  $P^*$ , is the set  $P$  equipped with the *dual order* defined by  $x \leq y$  in  $P^*$  if and only if  $y \leq x$  in  $P$ . Let  $P$  and  $Q$  be two posets. A map  $f : P \rightarrow Q$  is *order-preserving* if  $x \leq y$  in  $P$  implies  $f(x) \leq f(y)$  in  $Q$ ; this is an *embedding* if  $x \leq y$  in  $P$  is equivalent to  $f(x) \leq f(y)$  in  $Q$ ; if, in addition,  $f$  is onto, then this is an *order-isomorphism*. We say that  $P$  *embeds* into  $Q$  if there is an embedding from  $P$  into  $Q$ , a fact we denote  $P \leq Q$ ; if  $P \leq Q$  and  $Q \leq P$  then  $P$  and  $Q$  are *equimorphic*, a fact we denote  $P \equiv Q$ ; if there is an order-isomorphism from  $P$  onto  $Q$  we say that  $P$  and  $Q$  are *isomorphic* or have the same *order-type*, a fact we denote  $P \cong Q$ . We denote  $\omega$  the order type of  $\mathbb{N}$ , the set of natural integers,  $\omega^*$  the order type of the set of negative integers and  $\eta$  the order type of  $\mathbb{Q}$ , the set of rational numbers. A poset  $P$  is *well-founded* if every non-empty subset  $X$  of  $P$  has a minimal element. With the Axiom of dependent choice, this amounts to the fact that the order type  $\omega^*$  of the negative integers does not embed into  $P$ . If furthermore,  $P$  has no infinite antichain then  $P$  is *well-quasi-ordered*, w.q.o. in brief. A well-founded chain is *well-ordered*; its order type is an *ordinal*. A poset  $P$  is *scattered* if it does not embed the chain of rationals. We denote order types of chains and ordinals by greek letters  $\alpha, \alpha', \beta, \beta' \dots$ . If  $(P_i)_{i \in I}$  is a family of posets indexed by a poset  $I$ , the *lexicographic sum* of this family is the poset, denoted  $\sum_{i \in I} P_i$ , defined on the disjoint union of the  $P_i$ , that is formally the set of  $(i, x)$  such that  $i \in I$  and  $x \in P_i$ , equipped with the order  $(i, x) \leq (j, y)$  if either  $i < j$  in  $I$  or  $i = j$  and  $x \leq y$  in  $P_i$ . When  $I$  is the finite chain  $n := \{0, 1, \dots, n-1\}$  this sum is denoted  $P_0 + P_1 + \dots + P_{n-1}$ . When  $I := \omega$  this sum is denoted  $\sum_{i < \omega} P_i$  or  $P_0 + P_1 + \dots + P_n + \dots$ . We denote  $\sum_{i \in I}^* P_i$  for  $\sum_{i \in I^*} P_i$ ; when  $I := \omega$  we denote  $\sum_{i < \omega}^* P_i$  or  $\dots + P_n + \dots + P_1 + P_0$ . When  $I$  (resp.  $I^*$ ) is well ordered, or is an ordinal, we call *ordinal sum* (resp. *antiordinal sum*) instead of lexicographic sum. When all the  $P_i$  are equal to the same poset  $P$ , the lexicographic sum is denoted  $P.I$ , and called the *lexicographic product* of  $P$  and  $I$ . These definitions extend to order types, particularly to order types of chains (note that  $2\omega = \omega$  whereas  $\omega 2 = \omega + \omega$ ). Two order type  $\alpha$  and  $\beta$  are *equimorphic* if  $\alpha \leq \beta$  and  $\beta \leq \alpha$ , a fact we denote  $\alpha \equiv \beta$ . An order type  $\alpha$  is *indecomposable* if  $\alpha = \beta + \gamma$  implies  $\alpha \leq \beta$  or  $\alpha \leq \gamma$  (for exemple  $\eta$  is indecomposable); it is *right-indecomposable* if  $\alpha = \beta + \gamma$  with  $\gamma \neq 0$  implies  $\alpha \leq \gamma$ ; it is *strictly right-indecomposable* if  $\alpha = \beta + \gamma$  with  $\gamma \neq 0$  implies  $\beta < \alpha \leq \gamma$ . The *left-indecomposability* and the *strict left-indecomposability* are defined in the same way. The notions of indecomposability and strict right or left-indecomposability are preserved under equimorphy (but not the right or the left-indecomposability). A sequence  $(\alpha)_{n < \omega}$  of order types is *quasi-monotonic* if  $\{m : \alpha_n \leq \alpha_m\}$  is infinite for each  $n$ , its sum  $\alpha = \sum_{n < \omega} \alpha_n$  is right-indecomposable or equivalently  $\alpha' = \sum_{n < \omega}^* \alpha_n$  is left-indecomposable. If an order type  $\alpha$  can be written under the form  $\alpha = \alpha_0 + \alpha_1 + \dots + \alpha_{n-1}$  with each  $\alpha_i$  indecomposable, then it is a *decomposition* of  $\alpha$  of *length*  $n$ . This decomposition is *canonical* if its length is

minimal. An order type  $\alpha$  is *indivisible* if for every partition of a chain  $A$  of order type  $\alpha$  into  $B \cup C$ , then either  $A \leq B$  or  $A \leq C$ . An indivisible order type is indecomposable, an indecomposable scattered order type distinct of 0 and 1 is strictly right-indecomposable or strictly left-indecomposable (indecomposable, resp. indivisible, order types are said additively indecomposable, resp. indecomposable, by J.G.Rosenstein [38]).

The following theorem rassembles the fundamental properties of scattered order types (cf. the exposition given by J.G.Rosenstein [38]).

**Theorem 3.3.** (i) *Every countable order type is scattered or equimorphic to  $\eta$ .* (G. Cantor, 1897 ).  
(ii) *The class  $\mathcal{D}$  of scattered order types is the smallest class of order types containing 0, 1 and stable by ordinal and antiordinal sum.* (F. Hausdorff, 1908).  
(iii) *The class  $\mathcal{D}$  is well-quasi-ordered under embeddability.* (R. Laver, 1971).  
(iv) *For every countable scattered order type  $\alpha$ , the set  $\mathcal{D}(\alpha)$  of order types  $\beta$  which embed into  $\alpha$ , considered up to equimorphy, is at most countable.* (R. Laver, 1971).

An easy consequence of (iii) of Theorem 3.3 above is this:

**Corollary 3.2.** (R.Laver, 1971) (i) *Every scattered order type is a finite sum of indecomposable order types and has a partition into finitely many indivisible order types.*  
(ii) *The class of indecomposable countable scattered order types is the smallest class of order types containing 0, 1 and stable by  $\omega$  and  $\omega^*$  sums of quasi-monotonic sequences.*

We give, without proof, two, more technical, results we need.

**Lemma 3.1.** *Let  $\alpha$  be a scattered order type distinct from 0 and 1.*

(i) *The set  $\text{Idiv}(\alpha)$  of indivisible order types  $\beta$  which embeds into  $\alpha$  is a finitely generated initial segment of the collection of indivisible order types;*  
(ii) *let  $\nu$  be a maximal member of  $\text{Idiv}(\alpha)$  then  $\nu + 1$  (resp.  $1 + \nu$ ) does not embed into  $\alpha$  and  $\nu$  is strictly right (resp.left)-indecomposable if  $\alpha$  is strictly right (resp.left)-indecomposable.*

**Lemma 3.2.** *Let  $\alpha$  be a countable scattered order type. If  $\alpha$  is strictly right-indecomposable then  $\alpha = \Sigma_{\lambda < \mu} \alpha_\lambda$  where  $\mu$  is an indecomposable ordinal, every  $\alpha_\lambda$  is strictly left-indecomposable and verifies  $\alpha_\lambda < \alpha$ .*

**2.2. Initial segments and ideals of a poset.** Let  $P$  be a poset, a subset  $I$  of  $P$  is an *initial segment* of  $P$  if  $x \in P$ ,  $y \in I$  and  $x \leq y$  imply  $x \in I$ . If  $A$  is a subset of  $P$ , then  $\downarrow A = \{x \in P : x \leq y \text{ for some } y \in A\}$  denotes the least initial segment containing  $A$ . If  $I = \downarrow A$  we say that  $I$  is *generated* by  $A$  or  $A$  is *cofinal* in  $I$ . If  $A = \{a\}$  then  $I$  is a *principal initial segment* and we write  $\downarrow a$  instead of  $\downarrow \{a\}$ . A *final segment* of  $P$  is any initial segment of  $P^*$ . We denote by  $\uparrow A$  the final segment

generated by  $A$ . If  $A = \{a\}$  we write  $\uparrow a$  instead of  $\uparrow \{a\}$ . A subset  $I$  of  $P$  is *up-directed* if every pair of elements of  $I$  has a common upper-bound in  $I$ . An *ideal* is a non empty up-directed initial segment of  $P$  (in some other texts, the empty set is an ideal). We denote  $I(P)$ , resp.  $I_{<\omega}(P)$ , resp.  $J(P)$ , the set of initial segments, resp. finitely generated initial segments, resp. ideals of  $P$  ordered by inclusion and we set  $J_*(P) := J(P) \cup \{\emptyset\}$ ,  $I_0(P) := I_{<\omega}(P) \setminus \{\emptyset\}$ . We note that  $I_{<\omega}(P)$  is the set of compact elements of  $I(P)$ , hence  $J(I_{<\omega}(P)) \cong I(P)$ . Moreover  $I_{<\omega}(P)$  is a lattice and in fact a distributive lattice, if and only if,  $P$  is  $\downarrow$ -closed that is the intersection of two principal initial segments of  $P$  is a finite union, possibly empty, of principal initial segments.

We will need the following facts.

**Lemma 3.3.** *Let  $P$  be a poset and  $x \in P$ ; then, once ordered by inclusion, the set  $J_x(P) := \{J \in J(P) : x \in J\}$  is order-isomorphic to  $J(P \cap \uparrow x)$ .*

**Proof.** Let  $\psi : J_x(P) \rightarrow J(P \cap \uparrow x)$  be defined by  $\psi(J) := J \cap \uparrow x$ . This map is clearly order-preserving. The fact that it is an embedding holds on the remark that if an ideal  $J$  of  $P$  contains  $x$  then  $J \cap \uparrow x$  is cofinal in  $J$ . Indeed, let  $I, J \in J_x(P)$  such that  $\psi(I) \subseteq \psi(J)$ ; then  $I = \downarrow \psi(I) \subseteq \downarrow \psi(J) = J$ .  $\square$

Let  $P$  and  $Q$  be two posets, the *direct product* of  $P$  and  $Q$  denoted  $P \times Q$  is the set of  $(p, q)$  for  $p \in P$  and  $q \in Q$ , equipped with the product order; that is  $(p, q) \leq (p', q')$  if  $p \leq p'$  and  $q \leq q'$ . The *direct sum* of  $P$  and  $Q$  denoted  $P \oplus Q$  is the disjoint union of  $P$  and  $Q$  with no comparability between the elements of  $P$  and the elements of  $Q$  (formally  $P \oplus Q$  is the set of couples  $(x, 0)$  with  $x \in P$  and  $(y, 1)$  with  $y \in Q$  equipped with the order  $(p, q) \leq (p', q')$  if  $p \leq p'$  and  $q = q'$ ).

**Lemma 3.4.** *Let  $A_1, A_2$  be two posets, then  $J(A_1 + A_2) \cong J(A_1) + J(A_2)$ ,  $J(A_1 \times A_2) \cong J(A_1) \times J(A_2)$  and  $I(A_1 \oplus A_2) \cong I(A_1) \times I(A_2)$ .*

**Lemma 3.5.** *If an indivisible chain embeds into a product  $A_1 \times A_2$  then it embeds into  $A_1$  or into  $A_2$ .*

**Proof.** Let  $Q$  be a chain such that  $Q \leq A_1 \times A_2$ . Then  $Q \leq A'_1 \times A'_2$  where  $A'_i$  is a chain for  $i = 1, 2$ . We have  $J(Q) \leq J(A'_1 \times A'_2)$ . From Lemma 3.4,  $J(A'_1 \times A'_2) \cong J(A'_1) \times J(A'_2)$ . Since  $I(A'_i) = 1 + J(A'_i)$ , we have  $I(Q) \leq I(A'_1) \times I(A'_2)$ . From Lemma 3.4,  $I(A'_1) \times I(A'_2) \cong I(A'_1 \oplus A'_2)$ . A maximal chain of  $I(A'_1 \oplus A'_2)$  extending  $I(Q)$  is of the form  $I(C)$  where  $C$  is a chain extending the order on  $A'_1 \oplus A'_2$ . Since  $Q \leq C$ ,  $Q = B_1 \cup B_2$  with  $B_i \leq A'_i$  for  $i = 1, 2$ . Hence, if  $Q$  is indivisible,  $Q \leq A'_i$ , hence  $Q \leq A_i$ , for some  $i \in \{1, 2\}$ .  $\square$

**2.3. Join-semilattices.** A *join-semilattice* is a poset  $P$  such that arbitrary elements  $x, y$  have a join that we denote  $x \vee y$ . We denote  $\mathbb{J}$  the class of join-semilattices having a least element. If  $P \in \mathbb{J}$  then, since  $J(P)$  is an algebraic lattice, every maximal chain  $C$  is an algebraic lattice too, hence is of the form  $I(D)$  where  $D$  is some chain. Given an order type  $\alpha$ , let  $\mathbb{J}_\alpha$  be the class of  $P \in \mathbb{J}$  such that  $J(P)$  contains a chain of type  $I(\alpha)$  and let  $\mathbb{J}_{-\alpha} := \mathbb{J} \setminus \mathbb{J}_\alpha$ . Let  $\mathbb{B}$  be a subset of  $\mathbb{J}$ , let  $\uparrow \mathbb{B}$  be the

class of  $P \in \mathbb{J}$  such that  $P$  contains as a join-semilattice a member of  $\mathbb{B}$  and set  $Forb_{\mathbb{J}}(\mathbb{B}) := \mathbb{J} \setminus \uparrow \mathbb{B}$ . It is natural to ask whether for every countable order type  $\alpha$ , there is some finite  $\mathbb{B}$  such that  $\mathbb{J}_{-\alpha} = Forb_{\mathbb{J}}(\mathbb{B})$ . Looking at the class  $\mathbb{J}'$  of  $P \in \mathbb{J}$  such that  $J(P) \in \mathbb{L}$  we are just able to provide many  $\alpha$ 's such that:

$$\mathbb{J}'_{-\alpha} = Forb_{\mathbb{J}'}(\{1 + \alpha, [\omega]^{<\omega}\})$$

**Lemma 3.6.** *Let  $P, Q$  be two join-semilattices with a zero. Then:*

- (i)  *$Q$  embeds into  $P$  as a join-semilattice iff  $Q$  embeds into  $P$  as a join-semilattice, with the zero preserved.*
- (ii) *If  $Q$  embeds into  $P$  as a join-semilattice, then  $J(Q)$  embeds into  $J(P)$  by a map preserving arbitrary joins.*
- Suppose  $Q := I_{<\omega}(R)$  for some poset  $R$ , then:*
- (iii)  *$Q$  embeds into  $P$  as a poset iff  $Q$  embeds into  $P$  as a join-semilattice.*
- (iv)  *$J(Q)$  embeds into  $J(P)$  as a poset iff  $J(Q)$  embeds into  $J(P)$  by a map preserving arbitrary joins.*
- (v) *If  $\downarrow x$  is finite for every  $x \in R$  then  $Q$  embeds into  $P$  as a poset iff  $J(Q)$  embeds into  $J(P)$  as a poset.*

**Proof.** (i) Let  $f : Q \rightarrow P$  satisfying  $f(x \vee y) = f(x) \vee f(y)$  for all  $x, y \in Q$ . Set  $g(x) := f(x)$  if  $x \neq 0$  and  $g(0) := 0$ . Then  $g$  preserves arbitrary finite joins.  
(ii) Let  $f : Q \rightarrow P$  and  $\bar{f} : J(Q) \rightarrow J(P)$  defined by  $\bar{f}(I) := \downarrow \{f(x) : x \in I\}$ . If  $f$  preserve finite joins, then  $\bar{f}$  preserves arbitrary joins.  
(iii) Let  $f : Q \rightarrow P$ . Taking account that  $Q := I_{<\omega}(R)$ , set  $g(\emptyset) := 0$  and  $g(I) := \bigvee \{f(\downarrow x) : x \in I\}$  for each  $I \in I_{<\omega}(R) \setminus \{\emptyset\}$ .  
(iv) Let  $f$  be an embedding from  $J(Q)$  into  $J(P)$ . Taking account that  $J(Q) \cong I(R)$ , set  $g(I) := \bigvee \{f(\downarrow x) : x \in I\}$  for each  $I \in I(R) \setminus \{\emptyset\}$  and  $g(\emptyset) := 0$ . Note that  $g(\downarrow x) = f(\downarrow x)$  for all  $x \in I$  and  $g(I) \subseteq f(I)$  for all  $I \in I(R)$ . Suppose  $I \notin J$ . Let  $x \in I \setminus J$ . Since  $x \in I$ , we have  $f(\downarrow x) \notin f(J)$ . Since  $g(J) \subseteq f(J)$  we have  $f(\downarrow x) \notin g(J)$ . Hence  $g(I) \not\subseteq g(J)$ .  $\square$

The fact that a join-semilattice  $P$  contains a join-subsemilattice isomorphic to  $[\omega]^{<\omega}$  amounts to the existence of an infinite independent set. Let us recall that a subset  $X$  of a join-semilattice  $P$  is *independent* if  $x \not\leq \bigvee F$  for every  $x \in X$  and every non empty finite subset  $F$  of  $X \setminus \{x\}$ .

**Theorem 3.4.** [8] [25] *Let  $\kappa$  be a cardinal number; for a join-semilattice  $P$  the following properties are equivalent:*

- (i)  *$P$  contains an independent set of size  $\kappa$ ;*
- (ii)  *$P$  contains a join-subsemilattice isomorphic to  $[\kappa]^{<\omega}$ ;*
- (iii)  *$P$  contains a subposet isomorphic to  $[\kappa]^{<\omega}$ ;*
- (iv)  *$J(P)$  contains a subposet isomorphic to  $\mathfrak{P}(\kappa)$ ;*
- (v)  *$\mathfrak{P}(\kappa)$  embeds into  $J(P)$  via a map preserving arbitrary joins.*

**Proposition 3.1.** *Let  $P$  be a poset. The following properties are equivalent:*

- (i)  *$P$  contains an antichain of size  $\kappa$ ;*

- (ii)  $I_{<\omega}(P)$  contains a subset isomorphic to  $[\kappa]^{<\omega}$  ;
- (iii)  $\mathfrak{P}(\kappa)$  embeds into  $I(P)$ .

### 3. The class $\mathbb{K}$ and a proof of Theorem 3.1

**3.1. Property  $(p_1)$ .** The fact  $(p_1)$  holds is obvious.

**3.2. Properties  $(p_2)$  and  $(p_3)$ .**

**Lemma 3.7.** *Let  $\alpha$  and  $\beta$  be two order types such that  $\alpha + 1 \in \mathbb{K}$ . If  $P \in \mathbb{L}_{\alpha+1+\beta}$  and  $P$  contains no infinite independent set then  $P$  contains an element  $x$  such that  $1 + \alpha + 1$  embeds into  $P' := \downarrow x$  and  $P'' := \uparrow x \in \mathbb{L}_\beta$ .*

**Proof.** Let  $C$  be a chain of type  $\alpha + 1 + \beta$ , let  $c \in C$  such that the type of  $A := \downarrow c$  is  $\alpha + 1$  and the type of  $B := \{y : c < y\}$  is  $\beta$ . Let  $\varphi$  be an embedding from  $I(C)$  into  $J(P)$  and let  $P_1 := \varphi(A)$ . Since  $P_1 \in J(P)$  then  $J(P_1)$  is a sublattice of  $J(P)$ , and since  $J(P_1)$  embeds  $I(\alpha + 1)$ ,  $P_1 \in \mathbb{L}_{\alpha+1}$ . Since  $P_1$  contains no infinite independent set and  $\alpha + 1 \in \mathbb{K}$ , then  $1 + \alpha + 1$  embeds into  $P_1$ . Let  $f$  be an embedding from  $A$  into  $P_1$ ,  $x := f(c)$  and  $P' := \downarrow x$ . Clearly  $1 + \alpha + 1 \leq P'$ . Let  $P'' := \uparrow x$ . Clearly  $P'' \in \mathbb{L}$ . According to Lemma 3.3,  $J(P'') \cong J_x(P)$ ; since trivially  $I(\beta) \leq J_x(P)$ , we get  $P'' \in \mathbb{L}_\beta$ , as required.  $\square$

**3.2.1. Proof of Property  $(p_2)$ .** Let  $\alpha$  and  $\beta$  be two order types such that  $\alpha + 1, \beta \in \mathbb{K}$ . Let  $P \in \mathbb{L}_{\alpha+1+\beta}$ . If  $P$  contains no infinite independent set then, from Lemma 3.7,  $P$  contains an element  $x$  such that  $1 + \alpha + 1$  embeds into  $P' := \downarrow x$  and  $P'' := \uparrow x \in \mathbb{L}_\beta$ . Since  $\beta \in \mathbb{K}$  and  $P'' \in \mathbb{L}_\beta$  then  $1 + \beta$  embeds into  $P''$ , hence  $1 + \alpha + 1 + \beta$  embeds into  $P$ .  $\square$

**3.2.2. Proof of Property  $(p_3)$ .** Let  $\gamma := \sum_{n < \omega} (\alpha_n + 1)$  and let  $\beta_n := \sum_{m \geq n} (\alpha_m + 1)$  for  $n < \omega$ . Let  $P \in \mathbb{L}_\gamma$ . Suppose that  $P$  has no infinite independent set. We construct an infinite increasing sequence  $x_0, x_1, \dots, x_n, \dots$  of elements of  $P$  such that, for each  $k < \omega$ , we have  $1 + \alpha_k + 1 \leq Q_k$  and  $I(\beta_k) \leq J(P_k)$  where  $Q_k$  is the interval  $[x_k, x_{k+1}]$  and  $P_k := \uparrow x_k$ . Put  $x_0 := 0$ ,  $P_0 := \uparrow x_0$ . Since  $P_0 = P$  and  $\beta_0 = \gamma$ , we have  $I(\beta_0) \leq J(P_0)$ . Suppose  $x_0, \dots, x_n$  constructed. We have  $I(\beta_n) \leq J(P_n)$ . Since  $\beta_n = \alpha_n + 1 + \beta_{n+1}$  and  $\alpha_n + 1 \in \mathbb{K}$ , Lemma 3.7 asserts that the semilattice  $P_n$  contains an element  $x_{n+1}$  such that  $1 + \alpha_n + 1 \leq Q_n$  and  $I(\beta_{n+1}) \leq J(P_{n+1})$  for  $Q_n := [x_n, x_{n+1}]$ ,  $P_{n+1} := \uparrow x_{n+1}$ . Since  $1 + \alpha_n + 1$  embeds into  $[x_n, x_{n+1}]$  for all  $n < \omega$  it follows that  $1 + \gamma = 1 + \sum_{n < \omega} (\alpha_n + 1)$  embeds into  $P$ .

**3.3. Property  $(p_4)$ .** The fact that  $\omega^* \in \mathbb{K}$  is a consequence of Theorem 1.3 [9]. We use similar ingredients for Property  $(p_4)$ . Given a join-semilattice  $P$ ,  $x \in P$  and  $J \in J(P)$  we denote  $\downarrow x \vee J$ , as well as  $\{x\} \vee J$ , the join - in  $J(P)$ - of  $\downarrow x$  and  $J$ . We say that a non-empty chain  $\mathcal{I}$  of ideals of  $P$  is *separating* if for every  $I \in \mathcal{I} \setminus \{\cup \mathcal{I}\}$ , every  $x \in \cup \mathcal{I} \setminus I$ , there is some  $J \in \mathcal{I}$  such that  $I \not\subseteq \{x\} \vee J$ . We recall the following fact (Lemma 3.1 [9]). For reader's convenience we give the proof.

**Lemma 3.8.** *A join-semilattice  $P$  contains an infinite independent set if and only if it contains an infinite separating chain of ideals.*



**Proof.** Let  $\mathcal{I}$  be a separating chain of ideals. Define inductively an infinite sequence  $x_0, I_0, \dots, x_n, I_n, \dots$  such that  $I_0 \in \mathcal{I} \setminus \{\cup \mathcal{I}\}$ ,  $x_0 \in \cup \mathcal{I} \setminus I_0$  and such that:

$a_n$ )  $I_n \in \mathcal{I}$ ;

$b_n$ )  $I_n \subset I_{n-1}$ ;

$c_n$ )  $x_n \in I_{n-1} \setminus (\{x_0 \vee \dots \vee x_{n-1}\} \vee I_n)$  for every  $n \geq 1$ .

The construction is immediate. Indeed, since  $\mathcal{I}$  is infinite then  $\mathcal{I} \setminus \{\cup \mathcal{I}\} \neq \emptyset$ . Choose arbitrary  $I_0 \in \mathcal{I} \setminus \{\cup \mathcal{I}\}$  and  $x_0 \in \cup \mathcal{I} \setminus I_0$ . Let  $n \geq 1$ . Suppose  $x_k, I_k$  defined and satisfying  $a_k), b_k), c_k)$  for all  $k \leq n-1$ . Set  $I := I_{n-1}$  and  $x := x_0 \vee \dots \vee x_{n-1}$ . Since  $I \in \mathcal{I}$  and  $x \in \cup \mathcal{I} \setminus I$ , there is some  $J \in \mathcal{I}$  such that  $I \not\subseteq \{x\} \vee J$ . Let  $z \in I \setminus (\{x\} \vee J)$ . Set  $x_n := z$ ,  $I_n := J$ . The set  $X := \{x_n : n < \omega\}$  is independent. Indeed if  $x \in X$  then since  $x = x_n$  for some  $n$ ,  $n < \omega$ , condition  $c_n$ ) asserts that there is some ideal containing  $X \setminus \{x\}$  and excluding  $x$ .  $\square$

**Lemma 3.9.** *Let  $\alpha$  be a countable order type and  $P \in \mathbb{J}_\alpha$  satisfying the following conditions:*

- 1) *For every  $x$  in  $P$ , the chain  $I(\alpha)$  does not embed into  $J(\downarrow x)$ ;*
- 2) *For every  $a \in \alpha$ , the order type  $\alpha$  embeds into  $\downarrow a$ ;*
- 3) *The lattice  $L_{J(\beta)}$  does not embed into  $J(P)$  as a sublattice for  $\beta := \alpha$  if  $\alpha = \eta$  and  $\beta \in \{\omega, \omega^*\}$  otherwise.*

*Then  $P$  contains an infinite independent set.*

**Proof.** Let  $\mathcal{C} \subseteq J(P)$  be a chain isomorphic to  $I(\alpha)$  and  $\mathcal{C}^* := \mathcal{C} \setminus \{I_*\}$ , where  $I_*$  is the least element of  $\mathcal{C}$ . According to Lemma 3.8 it suffices to prove that  $\mathcal{C}^*$  is separating. Suppose it is not. Let  $I_0 \in \mathcal{C}^* \setminus \cup \mathcal{C}^*$  and  $x_0 \in \cup \mathcal{C}^* \setminus I_0$  witnessing it. Set  $\mathcal{C}_0^* := \{I \in \mathcal{C}^* : I \subseteq I_0\}$ . For every  $I \in \mathcal{C}_0^*$  we have  $(\downarrow x_0) \vee I = (\downarrow x_0) \vee I_0$ . Since  $I_0$  is non-empty then, from condition 2),  $\mathcal{C}_0^*$  contains a subset  $\mathcal{A}$  isomorphic to  $J(\alpha)$ . Let  $\mathcal{B}$  be the image of  $\mathcal{A} \cup \{I_*\}$  by the map  $\phi : J(P) \rightarrow J(\downarrow x_0)$  defined by  $\phi(I) := I \cap (\downarrow x_0)$  for every  $I \in J(P)$ . The chain  $\mathcal{A} \cup \{I_*\}$  is isomorphic to  $I(\alpha)$ , whereas its image  $\mathcal{B}$  does not embed  $I(\alpha)$  because of condition 1). Since  $\alpha$  is countable and, from condition 2), indecomposable, it follows that  $\phi$  is constant on a subset  $\mathcal{D}$  of  $\mathcal{A}$  which is isomorphic to  $J(\beta)$ , with  $\beta := \alpha$  if  $\alpha = \eta$  and  $\beta \in \{\omega, \omega^*\}$  otherwise (if  $\alpha = \eta$  then  $\phi$  cannot be one-to-one on a subset of  $\mathcal{A}$  isomorphic to  $\alpha$ , whereas if  $\alpha$  is scattered then, being indecomposable, it is equimorphic to  $J(\alpha)$  and, as it is easy to see, a map from  $\alpha$  on a strictly smaller order type is constant on an infinite subset, hence on a subset of type  $\omega$  or  $\omega^*$ ). The sublattice of  $J(P)$  generated by  $\downarrow x$  and  $\mathcal{D}$  is isomorphic to  $L_{J(\beta)}$ .  $\square$

This, added to a very simple trick, gives  $\omega^* \in \mathbb{K}$  and, more precisely:

**Theorem 3.5.** *If  $L_{\omega^*}$  does not embed, as a lattice, into the lattice  $J(P)$  of ideals of a join-semilattice  $P$ , then  $J(P)$  is well-founded if and only if  $P$  is well-founded and has no infinite independent set.*

**Proof.** Suppose  $P$  well-founded but  $J(P)$  not well-founded. Then  $P$  contains an ideal  $P'$  such that  $J(P')$  is not well-founded but  $J(\downarrow x)$  is well-founded for every  $x \in P'$ . Indeed, if  $P$  itself is not suitable, select  $x$  minimal in  $P$  such that  $J(\downarrow x)$  is not well-founded. Every  $P' \in J(\downarrow x)$  such that  $J(P')$  is not well-founded will do.

From Lemma 3.9 above,  $P'$  contains an infinite independent set, hence  $P$  contains an infinite independent set.  $\square$

In the next lemma, we extend the scope of the trick used above, in the same vein as in [34], Lemma 3.4.7.

**Lemma 3.10.** *Let  $\alpha := \sum_{n < \omega}^* \alpha_n$  be an  $\omega^*$ -sum of non-zero order types which is left-indecomposable. If  $P \in \mathbb{J}_\alpha$  then either:*

(i)  *$P$  contains some ideal  $P' \in \mathbb{J}_\alpha$  such that for every  $x \in P'$ , the chain  $I(\alpha)$  does not embed into  $J(\downarrow x)$ ;*

*or*

(ii)  *$P$  contains an  $\omega^*$ -sum  $P'' := \sum_{n < \omega}^* P_n$  where  $P_n \in \mathbb{J}_{\alpha_n}$  is a convex subset of  $P$  for every  $n < \omega$ .*

**Proof.** Set  $E := \{x \in P : I(\alpha) \leq J(\downarrow x)\}$ .

**Case 1.** There is some  $J \in \mathbb{J}_\alpha \cap J(P)$  such that  $E \cap J = \emptyset$ . Set  $P' := J$ .

**Case 2.**  $E \cap J \neq \emptyset$  for every  $J \in \mathbb{J}_\alpha \cap J(P)$ .

**Claim.** For every  $\beta$ ,  $1 + \beta \leq \alpha$ , and every  $J \in \mathbb{J}_\alpha \cap J(P)$  there is some  $x \in E \cap J$  such that  $I(\beta) \leq J(J \uparrow x)$ .

**Proof of the Claim.** Let  $\mathcal{C} \subseteq J(J)$  be a chain isomorphic to  $I(\alpha)$  and  $I_*$  be the least element of  $\mathcal{C}$ . There is some  $J' \in J(J) \setminus \{J_*\}$  such that  $I(\beta) \leq \{J'' \in J(J) : J' \subseteq J''\}$ . Since  $\alpha$  is left-indecomposable,  $I(\alpha) \leq J(J')$ . Since we are in Case 2,  $E \cap J' \neq \emptyset$ . Let  $x \in E \cap J'$ . From Lemma 3.3 the poset  $J(J \uparrow x)$  is isomorphic to  $\{J'' \in J(J) : x \in J''\}$  which trivially embeds  $I(\beta)$ . This proves our claim.  $\square$

With the help of this claim, we construct a sequence

$$J_0, x_0, J_1, x_1, \dots, J_n, x_n, \dots$$

such that  $J_n \in J(P)$ ,  $x_n \in E \cap J_n$ ,  $I(\alpha_n) \leq J(J_n \uparrow x_n)$  and  $J_{n+1} \subseteq \downarrow x_n$  for every  $n < \omega$ . Indeed, set  $J_0 := P$ , choose  $x_0 \in E$  such that  $I(\alpha_0) \leq J(\uparrow x_0)$ . Suppose  $J_0, x_0, \dots, J_n, x_n$  constructed. Since  $x_n \in E \cap J_n$ , we may choose  $J_{n+1} \in J(\downarrow x_n)$  such that  $I(\alpha) \leq J(J_{n+1})$ . According to the claim above, we may choose  $x_{n+1} \in E \cap J_{n+1}$  such that  $I(\alpha_{n+1}) \leq J(J_{n+1} \uparrow x_{n+1})$ .

To conclude, set  $P_n := J_n \uparrow x_n$  and  $P'' := \sum_{n < \omega}^* P_n$ .  $\square$

**3.3.1. Proof of Property  $(p_4)$ .** Let  $\alpha := \sum_{n < \omega}^* \alpha_n$  satisfying the conditions of  $(p_4)$  and let  $P \in \mathbb{L}_\alpha$ . We need to show that either  $P$  contains a subchain isomorphic to  $1 + \alpha$  or an infinite independent set. Since for every integer  $n$ , the set  $\{m : \alpha_n \leq \alpha_m\}$  is infinite, the order type  $\alpha$  is left-indecomposable and we may apply Lemma 3.10.

**Case 1.**  $P$  contains some ideal  $P' \in \mathbb{J}_\alpha$  such that for every  $x \in P'$ , the chain  $I(\alpha)$  does not embed into  $J(\downarrow x)$ .

This says that  $P'$  satisfies condition 1) of Lemma 3.9. Since  $P \in \mathbb{L}_\alpha$ , then neither  $L_{\omega+1}$  nor  $L_{\omega^*}$  embeds into  $J(P)$  as a sublattice. The same holds for  $J(P')$ , hence  $P'$  satisfies condition 3) of Lemma 3.9. Since  $\alpha$  is left-indecomposable, condition 2) of this lemma is satisfied too. It follows then that  $P'$ , thus  $P$ , contains an infinite independent set.

**Case 2.**  $P$  contains an  $\omega^*$ -sum  $P'' := \sum_{n < \omega}^* P_n$  where  $P_n \in \mathbb{J}_{\alpha_n}$  is a convex subset of  $P$  for every  $n < \omega$ .

Let  $n < \omega$ . Clearly  $P_n \in \mathbb{L}$ , hence  $P_n \in \mathbb{L}_{\alpha_n}$ . Since  $\alpha_n \in \mathbb{K}$  then  $P_n$  contains either

a subchain isomorphic to  $1 + \alpha_n$  or an infinite independent set. If some  $P_n$  contains an infinite independent set, such a set is independent in  $P$  too. If not, then each  $P_n$  contains a subchain isomorphic to  $1 + \alpha_n$ , hence  $P''$  contains a chain of type  $\Sigma_{n < \omega}^*(1 + \alpha_n)$ . Since  $P$  has a least element, it contains a chain of type  $1 + \alpha$ .  $\square$

### 3.4. Property $(p_6)$ .

**Lemma 3.11.** *Let  $P \in \mathbb{J}_\eta$  then  $P$  contains either a copy of  $\eta$  or a convex subset  $P' \in \mathbb{J}_\eta$  such that  $\downarrow x \cap P' \notin \mathbb{J}_\eta$ , for every  $x \in P'$ .*

**Proof.** Set  $P \in \mathbb{J}_\eta^*$  if  $P \in \mathbb{J}_\eta$  and the second part of the above assertion does not hold.

**Claim** If  $P \in \mathbb{J}_\eta^*$  then  $P$  contains an element  $x$  such that  $P' := \downarrow x$  and  $P'' := \uparrow x$  belong to  $\mathbb{J}_\eta^*$ .

**Proof of the claim** let  $\mathcal{C} \subseteq J(P)$  be a chain isomorphic to  $I(\eta)$ . Let  $I \in \mathcal{C} \setminus X$  where  $X := \text{Min}(\mathcal{C}) \cup \text{Max}(\mathcal{C})$ . The set  $I$  is convex and belongs to  $\mathbb{J}_\eta$ . Since  $P \in \mathbb{J}_\eta^*$  there is  $x \in I$  such that  $I(\eta) \leq J(\downarrow x)$ . Put  $P' := \downarrow x$  and  $P'' := \uparrow x$ . From our choice,  $P' \in \mathbb{J}_\eta$ . Since  $x \in I \notin \text{Max}(\mathcal{C})$ , we have  $I(\eta) \leq J_x(P)$ , hence from Lemma 3.3,  $I(\eta) \leq J(P'')$ , that is  $P'' \in \mathbb{J}_\eta$ . If  $Q \in \mathbb{J}_\eta$  is a convex subset of  $P'$  or  $P''$ , this is a convex subset of  $P$  too, hence it contains some  $y$  such that  $\downarrow y \cap Q \in \mathbb{J}_\eta$ , proving that  $P', P'' \in \mathbb{J}_\eta^*$ , as required.  $\square$

From this, it follows easily that if  $P \in \mathbb{J}_\eta^*$ , the chain of the dyadic numbers of the interval  $]0, 1[$  embeds into  $P$ . Indeed, associate to each dyadic  $d$  a convex subset  $P_d \in \mathbb{J}_\eta^*$  and an element  $x_d \in P_d$  as follows. Put  $P_{1/2} := P$ . Let  $d := \frac{2m+1}{2n}$ , denote  $d' := \frac{4m+1}{2n+1}$ ,  $d'' := \frac{4m+3}{2n+1}$ . Suppose  $P_d$  defined. From the claim above, get an element, denoted  $x_d$ . Put  $P_{d'} := \downarrow x_d \cap P_d$ ,  $P_{d''} := \uparrow x_d \cap P_d$ . The correspondance  $d \rightarrow x_d$  is an embedding. Thus  $\eta \leq P$ .  $\square$

3.4.1. *Proof of Property  $(p_6)$ .* Let  $P \in \mathbb{L}_\eta$ . If  $\eta \not\leq P$  then, from Lemma 3.11,  $P$  contains a convex subset  $P' \in \mathbb{J}_\eta$ , such that Condition 1) of Lemma 3.9 is verified for  $\alpha := \eta$ . Condition 2) is trivially verified. Since  $P'$  is convex in  $P$ , then  $P' \in \mathbb{L}$ , hence  $L(\eta)$  does not embed in  $J(P')$  as a sublattice. Thus Condition 3) is verified too. Applying Lemma 3.9, we get that  $P'$ , hence  $P$ , contains an infinite independent set. Thus  $\eta \in \mathbb{K}$ .

### 3.5. Property $(p_5)$ .

**Lemma 3.12.** *Let  $\delta, \beta, \mu$  be three order types and  $\alpha := \delta + \beta + \mu$ . If  $\alpha \in \mathbb{K}$  and if, for every order type  $\beta'$ , the order type  $\beta$  embeds into  $\beta'$  whenever  $\alpha$  embeds into  $\alpha' := \delta + \beta' + \mu$ , then  $\beta \in \mathbb{K}$ .*

**Proof.** Let  $\beta_*$  obtained from  $\beta$  by taking out its first element, if any. We prove  $\beta_* \in \mathbb{K}$ . Properties  $(p_1)$  and  $(p_2)$  insure  $\beta \in \mathbb{K}$ . Let  $P \in \mathbb{L}_{\beta_*}$ . Suppose  $P$  contains no infinite independent set. Let  $Q := 1 + \delta + P + \mu$ . Then  $Q$  is a join-semilattice with 0 and with no infinite independent set. Moreover,  $Q \in \mathbb{L}_\alpha$ . Indeed, from  $J(Q) \cong 1 + J(\delta) + J(P) + J(\mu)$ , we get first  $I(\alpha) \leq Q$  that is  $Q \in \mathbb{J}_\alpha$  ( $I(\alpha) \cong 1 + J(\alpha) \cong 1 + J(\delta) + J(\beta) + J(\mu) \leq 1 + J(\delta) + I(\beta_*) + J(\mu) \leq 1 + J(\delta) + J(P) + J(\mu) \cong J(Q)$ ); next, we get that  $L(\lambda) \leq J(Q)$  implies  $L(\lambda) \leq J(P)$  for every order-type  $\lambda \neq 0$ , from which  $Q \in \mathbb{L}_\alpha$  follows. Since  $\alpha \in \mathbb{K}$  we have  $1 + \alpha \leq Q$ . Let  $\varphi$  be an order

isomorphism from  $1 + \alpha$  into  $Q (= 1 + \delta + P + \mu)$  and let  $\beta' := \varphi(\alpha) \cap P$ . The order type  $\alpha$  embeds into  $\delta + \beta' + \mu$  hence, from our hypothesis,  $\beta$  embeds into  $\beta'$ . Since  $\beta'$  is a subchain of  $P$ ,  $\beta$  embeds into  $P$ . Since  $P$  has a 0, this implies that  $1 + \beta_*$  embeds into  $P$ , hence  $\beta_* \in \mathbb{K}$  as required.  $\square$

**Lemma 3.13.** *Let  $\alpha \in \mathbb{K}$ . If  $\alpha$  has canonical form  $\alpha = \alpha_0 + \alpha_1 + \dots + \alpha_n$  then  $\alpha_p + \alpha_{p+1} + \dots + \alpha_q \in \mathbb{K}$  for  $0 \leq p \leq q \leq n$ .*

**Proof.** Let  $p, q$  so that  $0 \leq p \leq q \leq n$ . Set  $\delta := \alpha_0 + \alpha_1 + \dots + \alpha_{p-1}$ ,  $\beta := \alpha_p + \alpha_{p+1} + \dots + \alpha_q$  and  $\mu := \alpha_{q+1} + \dots + \alpha_n$ . Let  $\beta'$  be an order type such that  $\alpha$  embeds into  $\alpha' := \delta + \beta' + \mu$ . We prove  $\beta \leq \beta'$ . Lemma 3.12 insures  $\beta \in \mathbb{K}$ . Let  $A$  and  $A'$  be two chains having order type  $\alpha$  and  $\alpha'$  respectively; let us consider a partition of  $A$  in three intervals  $D, B, M$  of types  $\delta, \beta, \mu$  respectively, such that  $A = D + B + M$  and similarly consider a partition of  $A'$  in three intervals  $D', B', M'$  of types  $\delta, \beta', \mu$  respectively, such that  $A' = D' + B' + M'$ . Let  $\varphi$  be an embedding from  $A$  into  $A'$  and  $B_1 := \varphi^{-1}(B')$ . We show that  $\beta$  embeds into  $B_1$ , which implies that  $\beta$  embeds into  $\beta'$ . Indeed, since the decomposition of  $\alpha$  is canonical,  $\alpha_0 + \alpha_1 + \dots + \alpha_p \not\leq \alpha_0 + \alpha_1 + \dots + \alpha_{p-1}$ , hence  $\alpha_p \not\leq \varphi^{-1}(D') \cap B$ . Similarly  $\alpha_q \not\leq \varphi^{-1}(M') \cap B$ . But  $\alpha_p$  and  $\alpha_q$  are indecomposable, hence if we decompose  $B$  into intervals  $A_i$  of type  $\alpha_i$  such that  $B = A_p + \dots + A_q$  we obtain  $\alpha_p \leq A_p \setminus \varphi^{-1}(D')$  and  $\alpha_q \leq A_q \setminus \varphi^{-1}(M')$ . It follows that  $\beta = \alpha_p + \dots + \alpha_q$  embeds into  $B_1$ .  $\square$

**Lemma 3.14.** *Let  $\gamma := \alpha + \beta$  be a scattered order-type, where  $\alpha$  is infinite and strictly right-indecomposable,  $\beta$  indecomposable, and  $\gamma \not\leq \mathbb{K}$ . Then  $\gamma \notin \mathbb{K}$ .*

**Proof.** We construct a distributive lattice  $T$ , direct product of two chains, one with type  $1 + \alpha_0$  for some  $\alpha_0 \leq \alpha$ , the other of type  $1 + \beta$ , such that a)  $T$  contains no infinite independent set; b)  $I(\gamma) \leq J(T)$ ; c)  $1 + \gamma \not\leq T$ .

Since  $\alpha$  is infinite and strictly right-indecomposable then  $1 + \alpha \leq \alpha$ , hence  $1 + \gamma \leq \gamma$ . Let  $C$  be a chain of type  $1 + \gamma$  and  $A, B$  be a partition of  $C$  into an initial segment  $A$  of type  $1 + \alpha$  and a final segment  $B$  of type  $\beta$ . Let  $1 + B$  the chain obtained by adding to  $B$  a least element  $b_0 \notin B$  and let  $Q := A \times (1 + B)$ . We distinguish two cases.

**Case 1**  $\beta$  is strictly left-indecomposable. We set  $T := Q$ . We check that  $T$  satisfies properties a), b), c) above. For property a) this is trivial: as a product of two chains,  $T$  contains no independent set with three elements. For b) let  $\phi : J(C) \rightarrow J(T)$  defined by  $\phi(I) := I \times \{b_0\}$  if  $I \subseteq A$  and  $\phi(I) := A \times ((I \setminus A) \cup \{b_0\})$  otherwise. This is an embedding, hence  $I(\gamma) \leq J(T)$ . Finally for c), suppose for contradiction  $1 + \gamma \leq T$ . Let  $\varphi$  be an embedding from  $C$  into  $T$ . Let  $b \in B$ ,  $(x, y) := \varphi(b)$  and  $\alpha_1, 1 + \beta_1$  be the order types of  $\downarrow x$  and  $\downarrow y$  respectively; we have  $\alpha \leq \alpha_1 \times (1 + \beta_1)$ . Since  $\alpha$  is infinite, (i) of Lemma 3.1 applies: among the indivisible order types which embed into  $\alpha$ , there is a maximal one. Let  $\nu$  be such an order type. **Claim 1.**  $\nu \leq \beta$ . First, since  $\nu$  is indivisible and  $\nu \leq \alpha_1 \times (1 + \beta_1)$  then, from Lemma 3.5, either  $\nu \leq \alpha_1$  or  $\nu \leq 1 + \beta_1$ . Since  $\alpha$  is infinite and strictly right-indecomposable then  $\alpha_1 + 1 \leq \alpha$ . Thus, if  $\nu \leq \alpha_1$  then  $\nu + 1 \leq \alpha$  which is impossible from (ii) of Lemma 3.1. Hence,  $\nu \not\leq \alpha_1$ . Thus  $\nu \leq 1 + \beta_1$ . Since  $\nu$  is infinite then  $\nu \leq \beta$ , proving our claim. Now, from this claim,  $\beta$  is infinite and we can repeat what we did with  $\alpha$ .

Let  $\xi$  be maximal among the indivisible order types embedding  $\nu$  and which embed into  $\beta$ . The same arguments as above show **Claim 2.**  $\xi \leq \alpha$ . From these two claims, we get  $\nu \equiv \xi$  which is impossible since, from (ii) of Lemma 3.1,  $\nu$  and  $\xi$  must be respectively strictly right and strictly left-indecomposable. So  $1 + \gamma \not\leq T$ .

**Case 2**  $\beta$  strictly right-indecomposable. Because of Case 1, we may suppose  $\beta \neq 1$ . Let  $\mathcal{A}$  be the set of order types  $\alpha'$ ,  $\alpha' \leq \alpha$ , such that  $\alpha \leq \alpha' \times (1 + \beta')$  for some  $\beta'$  such that  $\beta' + 1 \leq \beta$ . This set is non empty (it contains  $\alpha$ ). Hence, it contains a minimal element w.r.t. embeddability (Theorem 3.3 (iii)). Let  $\alpha_0$  be such an element and  $A_0$  be a subchain of  $A$  of type  $1 + \alpha_0$ . Let  $T := A_0 \times (1 + B)$ . We check that  $T$  satisfies the properties a), b), c) above. This is trivial for a) and for the same reasons that in Case 1. For b) observe that since  $\alpha_0 \in \mathcal{A}$  there is an element  $b_0 \in B$  and an embedding  $\varphi$  from  $A$  into  $A_0 \times \{y \in 1 + B : y < b_0\}$ . Since  $\beta$  is strictly right-indecomposable, there is an embedding  $\theta$  from  $B$  into  $\{y \in 1 + B : y \geq b_0\}$ . Let  $\phi: J(C) \rightarrow J(T)$  defined by  $\phi(I) := \{z \in T : z \leq \varphi(x) \text{ for some } x \in I\}$  if  $I \subseteq A$  and  $\phi(I) := A_0 \times \{y \in 1 + B : y \leq \theta(x) \text{ for some } x \in I\}$  otherwise. This is an embedding, hence  $I(\gamma) \leq J(T)$ . For c) we prove  $\alpha + 1 \not\leq T$ . For a contradiction, suppose  $\alpha + 1 \leq T$ . Let  $A + 1$  be the chain obtained by adding to  $A$  a largest element  $a \notin A$ . Since  $1 + \alpha \leq \alpha$  there is an embedding  $\psi$  from  $A + 1$  into  $T$ . Let  $(x, y) := \psi(a)$ . Then  $\psi$  is an embedding from  $A + 1$  into  $\downarrow x \times \downarrow y$ . Let  $\alpha_1 := \alpha'_1 + 1$  be the type of  $\downarrow x$  and  $1 + \beta_1$  the type of  $\downarrow y$ . We have  $\alpha_1 \leq 1 + \alpha_0$  and  $1 + \alpha + 1 \leq \alpha_1 \times (1 + \beta_1)$ . Since  $1 + \alpha \leq \alpha$  then  $\alpha_1 \leq \alpha$  and since  $\beta$  is strictly right-indecomposable and distinct from 1, we have  $\beta_1 + 1 \leq \beta$ . Hence  $\alpha_1 \in \mathcal{A}$ . **Claim 3.**  $\alpha_0$  is strictly right-indecomposable and distinct from 1. Indeed, if  $\alpha_0 = 1$ , then  $\alpha \leq 1 + \beta'_1$  for some  $\beta'_1$  such that  $\beta'_1 + 1 \leq \beta$ . Since  $1 + \alpha \leq \alpha$ , we get  $\alpha \leq \beta'_1$  and since  $\beta$  is strictly right-indecomposable,  $\alpha + \beta \leq \beta$ , that is  $\gamma \leq \beta$ , which contradicts our assumption. Next, suppose  $\alpha_0 = \alpha'_0 + \alpha''_0$  with  $\alpha''_0 \neq 0$ . Since  $\alpha_0 \in \mathcal{A}$  there is some  $\beta'$  such that  $\beta' + 1 \leq \beta$  and  $\alpha \leq \alpha_0 \times (1 + \beta')$ . Moreover, since  $\alpha_0$  is minimal, we may suppose that for some embedding  $\varphi_0 : \alpha \rightarrow \alpha_0 \times (1 + \beta')$  the projection on  $\alpha_0$  is surjective. Since  $\alpha_0 \times (1 + \beta')$  decomposes into an initial segment of type  $\alpha'_0 \times (1 + \beta')$  and a final segment of type  $\alpha''_0 \times (1 + \beta')$ , then  $\varphi_0$  divides  $\alpha$  into  $\alpha'$  and  $\alpha''$  in such a way that  $\alpha' \leq \alpha'_0 \times (1 + \beta')$  and  $\alpha'' \leq \alpha''_0 \times (1 + \beta')$ . Since  $\alpha$  is strictly right-indecomposable and  $\alpha'' \neq 0$  then  $\alpha \leq \alpha''$ , hence  $\alpha''_0 \in \mathcal{A}$ . From the minimality of  $\alpha_0$  follows  $\alpha_0 \leq \alpha''_0$ , hence  $\alpha_0$  is strictly right-indecomposable as claimed. From this claim, the inequality  $\alpha_1 := \alpha'_1 + 1 \leq 1 + \alpha_0$  implies  $\alpha_1 < \alpha_0$ . Since  $\alpha_1 \in \mathcal{A}$ , this contradicts the minimality of  $\alpha_0$ .  $\square$

3.5.1. *Proof of Property (p<sub>5</sub>).* Let  $\alpha$  be a countable scattered order type. Suppose  $\alpha \in \mathbb{K}$ . From Corollary 3.2,  $\alpha$  has decomposition into finitely indecomposables order types, hence a canonical decomposition  $\alpha = \alpha_0 + \alpha_1 + \dots + \alpha_n$ . From Lemma 3.13,  $\alpha_i \in \mathbb{K}$  for all  $i \leq n$ . Let  $i < n$ . If  $\alpha_i$  is not strictly left-indecomposable then it is infinite and strictly right-indecomposable. Since the decomposition is canonical,  $\alpha_i + \alpha_{i+1}$  embeds neither into  $\alpha_i$  nor into  $\alpha_{i+1}$ . From Lemma 3.14,  $\alpha_i + \alpha_{i+1} \notin \mathbb{K}$ . This contradicts Lemma 3.13. Thus,  $\alpha$  has the form given in Property (p<sub>5</sub>). Conversely, suppose  $\alpha = \alpha_0 + \alpha_1 + \dots + \alpha_n$  with  $\alpha_i \in \mathbb{K}$ , for all  $i \leq n$ ,  $\alpha_i$  strictly left-indecomposable for  $i < n$  and  $\alpha_n$  indecomposable. If  $n = 0$  then  $\alpha = \alpha_0 \in \mathbb{K}$ .

If  $n > 0$ , suppose  $n$  minimal. Then  $\alpha_i \neq 0$  for all  $i < n$ , hence  $\alpha_i \equiv \alpha'_i + 1$  and  $\alpha \equiv \alpha' = \alpha'_0 + 1 + \alpha'_1 + 1 + \dots + \alpha'_{n-1} + 1 + \alpha_n$ . From Property  $(p_2)$ ,  $\alpha' \in \mathbb{K}$ . That is  $\alpha \in \mathbb{K}$ .

#### 4. The class $\mathbb{P}$ and a proof of Theorem 3.2

Define for each ordinal  $\xi$  the class  $\mathbb{P}_\xi$ , setting  $\mathbb{P}_0 := \{0, 1\}$ ,  $\mathbb{P}_\xi := \bigcup \{\mathbb{P}_{\xi'} : \xi' < \xi\}$  if  $\xi$  is a limit ordinal,  $\mathbb{P}_{\xi+1}$  equals to the set of order types  $\gamma$  which decompose into elements of  $\mathbb{P}_\xi$  under one of the forms stated in  $(p_2)$ ,  $(p_3)$  and  $(p_4)$  of the definition of  $\mathbb{P}$  (cf. Section 1). For an example,  $\mathbb{P}_1 = \mathbb{P}_0 \cup \{2, \omega, \omega^*\}$ ,  $\mathbb{P}_2 = \mathbb{P}_1 \cup \{3, 4, 1 + \omega^*, 2 + \omega^*, \omega^*2, \omega^{*2}, \omega^*n + \omega, n + \omega^*\omega, \omega\omega^* + n \text{ with } n < \omega\} \cup \{\gamma : \gamma \equiv (\omega^* + \omega)\omega^*\}$ . Since  $\mathbb{P}_0 \subseteq \mathbb{P}_1$  the sequence of  $\mathbb{P}_\xi$  is non decreasing and moreover  $\mathbb{P} = \mathbb{P}_{\omega_1}$ . Note that  $\mathbb{P}$  is not preserved under equimorphy (indeed  $\omega\omega^*\omega \notin \mathbb{P}$  while  $(\omega\omega^* + 1)\omega \in \mathbb{P}_3$ ). Let  $\gamma \in \mathbb{P}$ ; the  $\mathbb{P}$ -rank of  $\gamma$ , denoted  $\text{rank}_{\mathbb{P}}(\gamma)$ , is the least  $\xi$  such that  $\gamma \in \mathbb{P}_\xi$ . This notion of rank allows to prove properties of the elements of  $\mathbb{P}$  by induction on their rank, especially those given just below.

##### Claim 1

Let  $\gamma$  be a countable order type;

- 1.1) If  $\gamma \in \mathbb{P}$  then  $\delta \in \mathbb{P}$  for every initial segment  $\delta$  of  $\gamma$ ;
- 1.2)  $\gamma \in \mathbb{P}$  if and only if  $\gamma$  decomposes into a sum  $\gamma = \gamma_0 + \dots + \gamma_n$  of members of  $\mathbb{P}$ , each  $\gamma_i$  ( $i < n$ ) is strictly left-indecomposable and has a largest element,  $\gamma_n$  is indecomposable;
- 1.3) Suppose  $\gamma \in \mathbb{P}_{\xi+1} \setminus \{0, 1\}$  ( $\xi < \omega_1$ ). If  $\gamma$  is strictly right-indecomposable then  $\gamma = \sum_{n < \omega} (\alpha_n + 1)$  with  $\alpha_n + 1 \in \mathbb{P}_\xi$ ; if  $\gamma$  is strictly left-indecomposable then  $\gamma = \sum_{n < \omega}^* \alpha_n$  with  $\alpha_n \in \mathbb{P}_\xi$ , the  $\alpha_n$ 's forming a quasi-monotonic sequence.

##### Claim 2

The class of indecomposable members of  $\mathbb{P}$  is the smallest class  $\text{Ind}_{\mathbb{P}}$  of order types such that:

- 2.1)  $0, 1 \in \text{Ind}_{\mathbb{P}}$ ;
- 2.2) If  $(\alpha_n)_{n < \omega}$  is a quasi-monotonic sequence of elements of  $\text{Ind}_{\mathbb{P}}$  then  $\sum_{n < \omega}^* \alpha_n \in \text{Ind}_{\mathbb{P}}$ ; if, moreover, each  $\alpha_n$  has a largest element then  $\sum_{n < \omega} \alpha_n \in \text{Ind}_{\mathbb{P}}$ .

From Claim 2 and the fact that every countable indivisible scattered order-type is an  $\omega$ -sum or an  $\omega^*$ -sum of smaller indivisible order-types, we deduce:

##### Claim 3

If  $\gamma \in \mathbb{P}$  then each maximal indivisible order-type which embeds into  $\gamma$  is equimorphic to some element of  $\mathbb{P}$ .

**Lemma 3.15.** *Let  $\alpha$  be a countable, scattered and indecomposable, order type.*

- (i) *The set of  $\beta \in \mathbb{P}$  which embed into  $\alpha$  has a largest element (up to equimorphy) which we denote  $\mathbb{P}(\alpha)$ ;*
- (ii) *If  $\alpha \equiv \sum_{n < \omega}^* \alpha_n$  with each  $\alpha_n$  indecomposable and verifying  $0 < \alpha_n < \alpha$  then  $\mathbb{P}(\alpha) \equiv \sum_{n < \omega}^* \mathbb{P}(\alpha_n)$ .*
- (iii) *If  $\alpha \equiv \sum_{\lambda < \mu} \alpha_\lambda$  where  $\mu$  is an indecomposable ordinal, each  $\alpha_\lambda$  is strictly left-indecomposable and verifies  $0 < \alpha_\lambda < \alpha$  then  $\mathbb{P}(\alpha_\lambda) < \mathbb{P}(\alpha)$  for each  $\lambda < \mu$ .*

Moreover, there is a sequence  $\lambda_0 < \lambda_1 < \dots < \lambda_n < \dots$  such that  $\text{Sup}\{\lambda_n : n < \omega\} = \mu$  and  $\mathbb{P}(\alpha) \equiv \sum_{n < \omega} \mathbb{P}(\alpha_{\lambda_n})$ .

**Proof.** (i) We may suppose  $\alpha \neq 0, 1$ .

**Case1**  $\alpha$  is strictly right-indecomposable. Let  $\mathcal{A} := \{\gamma \in \mathbb{P} : \gamma \leq \alpha, \gamma = \beta + 1 \text{ for some } \beta\}$ . This set contains a countable subset  $X$  which is cofinal (cf. (iv) of Theorem 3.3 in Section 2). Let  $\beta_0 + 1, \beta_1 + 1, \dots, \beta_n + 1, \dots$  an enumeration of the elements of  $X$ , with possible repetitions, which is quasi-monotonic and let  $\beta := \beta_0 + 1 + \beta_1 + 1 + \dots + \beta_n + 1 + \dots$ . Since  $\mathbb{P}$  satisfies property  $(p_3)$ , we have  $\beta \in \mathbb{P}$ . Since  $\alpha$  is strictly right-indecomposable, we have  $\beta \leq \alpha$ . Let  $\gamma \in \mathbb{P}$  with  $\gamma \leq \alpha$ . Then, according to Claim 1.2,  $\gamma$  is a sum  $\gamma_0 + \gamma_1 + \dots + \gamma_n$  with  $\gamma_i$  strictly left-indecomposable with a largest element for  $i < n$ ,  $\gamma_n$  indecomposable and all  $\gamma_i \in \mathbb{P}$ . If  $\gamma_n$  is strictly left-indecomposable then, since from Claim 1.1 every initial segment belongs to  $\mathbb{P}$ ,  $\gamma$  embeds into a finite sum of elements of  $\mathcal{A}$ , while if  $\gamma_n$  is strictly right-indecomposable then, according to Claim 1.3, it is an  $\omega$ -sum of elements of  $\mathcal{A}$ , hence  $\gamma$  too. Since  $X$  is cofinal and the enumeration is quasi-monotonic then  $\gamma \leq \beta$ . Thus, up-to equimorphy,  $\beta$  is the largest element of  $\mathbb{P} \cap \downarrow \alpha$ .

**Case2**  $\alpha$  is strictly left-indecomposable. We consider the set  $\mathcal{B} := \{\gamma \in \mathbb{P} : \gamma \leq \alpha, \gamma = 1 + \beta\}$  and we proceed as in Case 1.

(ii) Suppose  $\alpha \equiv \sum_{n < \omega}^* \alpha_n$  with each  $\alpha_n$  indecomposable and  $0 < \alpha_n < \alpha$ . Since  $\alpha$  is indecomposable and  $0 < \alpha_0 < \alpha$ , then  $\alpha$  is strictly left-indecomposable. Since the  $\alpha_n$  are indecomposable and are distinct from 0 they form a quasi-monotonic sequence. But then, the sequence of the  $\mathbb{P}(\alpha_n)$  is quasi-monotonic too. Hence  $\gamma := \sum_{n < \omega}^* \mathbb{P}(\alpha_n)$  is indecomposable and belongs to  $\mathbb{P}$ , thus  $\gamma \leq \mathbb{P}(\alpha)$ . According to the proof of (i) above,  $\mathbb{P}(\alpha)$  is of the form  $\sum_{n < \omega}^* (1 + \beta_n)$  with  $1 + \beta_n \in \mathbb{P} \cap \downarrow \alpha$ . Let  $1 + \beta \in \mathbb{P} \cap \downarrow \alpha$ ; according to Claim 1.2, we may write  $1 + \beta = 1 + \gamma_1 + \dots + \gamma_m$  with each  $\gamma_i$  indecomposable belonging to  $\mathbb{P}$ . Each  $\gamma_i$  embeds into some  $\alpha_n$ , hence into  $\mathbb{P}(\alpha_n)$ . Because of the quasi-monotony of the sequence of the  $\mathbb{P}(\alpha_n)$  it follows  $1 + \beta \leq \sum_{n < \omega}^* \mathbb{P}(\alpha_n) := \gamma$ . Since  $\gamma$  is strictly left-indecomposable, this inequality applied to each  $1 + \beta_n$  leads to  $\mathbb{P}(\alpha) \leq \gamma$ . Finally,  $\mathbb{P}(\alpha) \equiv \sum_{n < \omega}^* \mathbb{P}(\alpha_n)$ .

(iii) Suppose  $\alpha = \sum_{\lambda < \mu} \alpha_\lambda$  with each  $\alpha_\lambda$  strictly left-indecomposable verifying  $0 < \alpha_\lambda < \alpha$ . From (ii) above, if  $\beta$  is a strictly left-indecomposable order type then  $\mathbb{P}(\beta)$  too; moreover, from Claim 1.1, we may suppose that  $\mathbb{P}(\beta)$  has a largest element. Hence, for every increasing sequence  $\lambda_0 < \lambda_1 < \dots < \lambda_n < \dots$  of members of  $\mu$ , we have  $\sum_{n < \omega} \mathbb{P}(\alpha_{\lambda_n}) \in \mathbb{P} \cap \downarrow \alpha$  proving  $\sum_{n < \omega} \mathbb{P}(\alpha_{\lambda_n}) \leq \mathbb{P}(\alpha)$ . Since  $\alpha$  is indecomposable and  $0 < \alpha_0 < \alpha$ , then  $\alpha$  is strictly right-indecomposable. According to the construction given in (i) above,  $\mathbb{P}(\alpha)$  is also strictly right-indecomposable and, in fact, is a sum  $\sum_{n < \omega} (\beta_n + 1)$  with  $\beta_n + 1 \in \mathbb{P}$ . From Claim 1.2 each of these  $\beta_n + 1$  is a finite sum of strictly left-indecomposable members of  $\mathbb{P}$ , hence  $\mathbb{P}(\alpha)$  is a sum  $\sum_{n < \omega} \gamma_n$  with  $\gamma_n$  strictly left-indecomposable and belonging to  $\mathbb{P}$ . Each  $\gamma_n$  embeds into some  $\alpha_\lambda$ ; indeed, since  $\gamma_n \leq \alpha$ , let  $\mu', \mu' \leq \mu$ , be the least ordinal such that  $\gamma_n \leq \sum_{\lambda < \mu'} \alpha_\lambda$ . Since  $\gamma_n$  is strictly left-indecomposable then  $\mu' := \mu'' + 1$  and then for the same reason  $\gamma_n \leq \alpha_{\mu''}$ . In fact, since  $\alpha$  is strictly right-indecomposable,  $\gamma_n$  embeds into cofinally many  $\alpha_\lambda$ 's, hence into cofinally many  $\mathbb{P}(\alpha_\lambda)$ 's. We can then

find a cofinal sequence  $\lambda_0 < \lambda_1 < \dots < \lambda_n < \dots$  such that  $\gamma_n \leq \mathbb{P}(\alpha_{\lambda_n})$ , proving  $\mathbb{P}(\alpha) \leq \sum_{n < \omega} \mathbb{P}(\alpha_{\lambda_n})$ . Finally, we have  $\mathbb{P}(\alpha) \equiv \sum_{n < \omega} \mathbb{P}(\alpha_{\lambda_n})$ , as claimed.  $\square$

**Proposition 3.2.** *For each countable scattered indecomposable order type  $\alpha$  there is a distributive lattice  $T_\alpha$  with 0 such that:*

- 1)  $T_\alpha$  contains no infinite independent set;
- 2) Each indivisible chain that embeds into  $T_\alpha$  embeds into  $\mathbb{P}(\alpha)$ ;
- 3)  $I(\alpha)$  embeds into  $J(T_\alpha)$ .

Thus if  $\alpha$  is indivisible and if  $\mathbb{P}(\alpha) < \alpha$  then  $T_\alpha$  reveals that  $\alpha \notin \mathbb{K}$ . This proves Theorem 3.2.

**Proof.** To each countable scattered indecomposable order-type  $\alpha$  we associate an ordered set  $P_\alpha$ , with a largest element if  $\alpha$  is strictly left-indecomposable, such that:

- 1')  $P_\alpha$  has no infinite antichain;
- 2')  $I_{<\omega}(P_\alpha)$  is a distributive lattice and every indivisible chain which embeds into  $I_{<\omega}(P_\alpha)$  embeds into  $\mathbb{P}(\alpha)$ ;
- 3') the order on  $P_\alpha$  extends to a linear order  $\leq_\alpha$  such that the chain  $\underline{P}_\alpha := (P_\alpha, \leq_\alpha)$  embeds  $\alpha$ .

Let  $T_\alpha := I_{<\omega}(P_\alpha)$ . Then  $T_\alpha$  satisfies conditions 1), 2) and 3). Indeed,  $J(T_\alpha) = J(I_{<\omega}(P_\alpha)) \cong I(P_\alpha)$ . Since  $P_\alpha$  has no infinite antichain,  $I(P_\alpha)$  does not embed  $\mathfrak{P}(\omega)$  (Proposition 3.1), hence, as it results from Theorem 3.4,  $T_\alpha$  has no infinite independent set, proving that 1) holds. Condition 2) is just Condition 2') in this case. Now, since  $\alpha \leq \underline{P}_\alpha$  it follows that  $I(\alpha) \leq I(\underline{P}_\alpha) \leq I(P_\alpha) \cong J(T_\alpha)$ , hence 3) holds.

Since the class of countable order-types is well-quasi-ordered, we may construct the  $P_\alpha$ 's by induction. Starting with  $\alpha = 1$  we put  $P_1 := 1$ . Next, we distinguish the following cases:

**Case 1:**  $\alpha$  strictly left-indecomposable. In this case  $\alpha \equiv \sum_{n < \omega}^* \alpha_n$  where the  $\alpha_n$ 's form a quasi-monotonic sequence of indecomposable order types ((ii) of Corollary 3.2). We put  $P_\alpha := (\sum_{n < \omega}^* P_{\alpha_n}) + 1$ . Conditions 1') and 3') are trivially satisfied. For 2') observe that from (ii) of Lemma 3.15 we have  $\mathbb{P}(\alpha) \equiv \sum_{n < \omega}^* \mathbb{P}(\alpha_n)$ .

**Case 2:**  $\alpha$  strictly right-indecomposable. From Lemma 3.2,  $\alpha$  writes  $\sum_{\lambda < \mu} \alpha_\lambda$  where  $\mu$  is an indecomposable ordinal, each  $\alpha_\lambda$  is strictly left-indecomposable and verifies  $\alpha_\lambda < \alpha$ . If  $\mu = \omega$  we put  $P_\alpha := \sum_{n < \omega} P_{\alpha_n}$ . In full generality, let  $Q$  be the set of integers ordered by the intersection of the natural order  $\leq_\omega$  on  $\omega$ , and the order  $\leq_\mu$  of type  $\mu$ , the order  $\leq_\mu$  been chosen such that 0 is the least element of  $Q$ . We put  $P_\alpha := \sum_{\lambda \in Q} P_{\alpha_\lambda}$ . Since the order of  $Q$  is the intersection of two well-orders,  $Q$  has no infinite antichain. Since the  $P_{\alpha_\lambda}$ 's have no infinite antichain, the sum  $\sum_{\lambda \in Q} P_{\alpha_\lambda}$  also. Hence 1') holds. By construction, the natural order on  $\mu$  is a linear extension of  $Q$ , thus the chain  $\underline{Q} := (Q, \leq_\mu)$  embeds  $\mu$ . From the inductive hypothesis, each  $P_{\alpha_\lambda}$  extends to a chain  $\underline{P}_{\alpha_\lambda}$  which embeds  $\alpha_\lambda$ . Hence  $\alpha := \sum_{\lambda < \mu} \alpha_\lambda \leq \underline{P}_\alpha := \sum_{\lambda \in Q} \underline{P}_{\alpha_\lambda}$  and 3') holds. It remains to prove that 2') holds. First,  $I_{<\omega}(P_\alpha)$  is a lattice. It suffices to check that if  $x, y \in P_\alpha$  then the initial segment  $Z := \downarrow x \cap \downarrow y$  is finitely generated. With no loss of generality, we may suppose that the  $P_{\alpha_\lambda}$ 's are pairwise disjoint and that  $P_\alpha$  is the union of them. If  $x$  and  $y$  are in the same  $P_{\alpha_\lambda}$  then  $Z = \downarrow (Z \cap P_{\alpha_\lambda}) \cup R_\lambda$  where  $R_\lambda := \cup \{P_{\alpha_{\lambda'}} : \lambda' < \lambda\}$ . From the inductive hypothesis,



$Z \cap P_{\alpha_\lambda}$  is a finitely generated initial segment of  $P_{\alpha_\lambda}$ , moreover, since each  $P_{\alpha_{\lambda'}}$  has a largest element  $1_{\alpha_{\lambda'}}$  and  $\{\lambda' \in Q : \lambda' < \lambda\}$  is finite, then  $R_\lambda$  is finitely generated, hence  $Z$  too. If  $x \in P_{\alpha_{\lambda'}}$ ,  $y \in P_{\alpha_{\lambda''}}$  with  $\lambda' \neq \lambda''$ , then  $Z = \cup \{P_{\alpha_\nu} : \nu \in \downarrow \lambda' \cap \downarrow \lambda''\}$ . Since each  $P_{\alpha_\nu}$  has a largest element  $1_{\alpha_\nu}$  and  $\downarrow \lambda' \cap \downarrow \lambda''$  is finite, then  $Z$  is finitely generated. Next, each indivisible chain  $C$  contained into  $I_{<\omega}(P_\alpha)$  embeds into  $\mathbb{P}(\alpha)$ . To see this, associate to each  $Z \in I_{<\omega}(P_\alpha)$  the set  $p(Z) := \{\lambda \in Q : Z \cap P_{\alpha_\lambda} \neq \emptyset\}$  and observe that this set is an initial segment of  $Q$ , hence is finite. Let  $C$  be a chain contained into  $I_{<\omega}(P_\alpha)$ . The set  $L$  formed by the  $p(Z)$  for  $Z \in C$  is a chain of  $I_{<\omega}(Q)$ . Hence, either  $L$  is finite or has type  $\omega$ . For each  $F \in L$  put  $C_F = \{Z \in C : p(Z) = F\}$ . Each  $C_F$  is an interval of  $C$  and  $C$  is the sum  $\sum_{F \in L} C_F$ . As a chain,  $C_F$  embeds into the direct product  $\prod_{\lambda \in \text{Max} F} I_{<\omega}(P_{\alpha_\lambda}) = \prod_{\lambda \in \text{Max} F} T_{\alpha_\lambda}$ . Let  $\gamma$  be an indivisible chain embedding into  $C$ . If  $\gamma$  embeds into one of the  $C_F$  then it embeds into  $\prod_{\lambda \in \text{Max} F} T_{\alpha_\lambda}$  hence into one of the  $T_{\alpha_\lambda}$  (cf. Lemma 3.5). From the inductive hypothesis, it embeds into  $\mathbb{P}(\alpha_\lambda)$  hence into  $\mathbb{P}(\alpha)$ . If not,  $L$  has type  $\omega$  and we can write  $\gamma = \sum_{n < \omega} \gamma_n$  with  $\gamma_n$  indivisible embedding into some  $C_{F_n}$ . We obtain  $\gamma \leq \sum_{n < \omega} \mathbb{P}(\alpha_{\lambda_n}) \leq \mathbb{P}(\alpha)$ .

□



## CHAPTER 4

### Infinite independent sets in distributive lattices

We show that a poset  $P$  contains a subset isomorphic to  $[\kappa]^{<\omega}$  if and only if the poset  $J(P)$  consisting of ideals of  $P$  contains a subset isomorphic to  $\mathcal{P}(\kappa)$ , the power set of  $\kappa$ . If  $P$  is a join-semilattice this amounts to the fact that  $P$  contains an independent set of size  $\kappa$ . We show that if  $\kappa := \omega$  and  $P$  is a distributive lattice, then this amounts to the fact that  $P$  contains either  $I_{<\omega}(\Gamma)$  or  $I_{<\omega}(\Delta)$  as sublattices, where  $\Gamma$  and  $\Delta$  are two special meet-semilattices already considered by J.D.Lawson, M.Mislove and H.A.Priestley.

#### 1. Presentation of the results

Let  $P$  be a poset. An *ideal* of  $P$  is a non-empty up-directed initial segment of  $P$ . The set  $J(P)$  of ideals of  $P$ , ordered by inclusion, is an interesting poset associated with  $P$ . And there are several results about their relationship, eg [11, 34]. If  $P := [\kappa]^{<\omega}$ , the set, ordered by inclusion, consisting of finite subsets of  $\kappa$ , then the poset  $J([\kappa]^{<\omega})$  is isomorphic to  $\mathcal{P}(\kappa)$ , the set, ordered by inclusion, consisting of arbitrary subsets of  $\kappa$ . We prove:

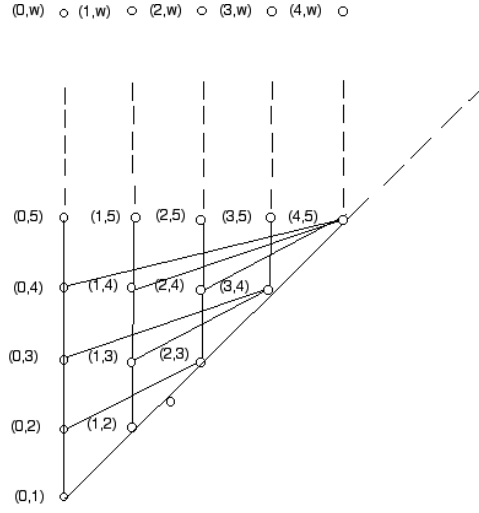
**Theorem 4.1.** *A poset  $P$  contains a subset isomorphic to  $[\kappa]^{<\omega}$  if and only if  $J(P)$  contains a subset isomorphic to  $\mathcal{P}(\kappa)$ .*

If  $P$  is a join-semilattice, then such containments amount to the existence of an independent set of size  $\kappa$ ; a subset  $X$  of  $P$  being *independent* if  $x \not\leq \bigvee F$  for every  $x \in X$  and every non-empty finite subset  $F$  of  $X \setminus \{x\}$ . The notion of independence is better understood in terms of closure operators. Let us recall that if  $\varphi$  is a closure operator on a set  $E$ , a subset  $X$  is *independent* if  $x \notin \varphi(X \setminus \{x\})$  for every  $x \in X$ . Also, if  $\varphi$  is algebraic then: a)  $\mathcal{F}_\varphi$ , the set of closed sets, is an algebraic lattice, b)  $\mathcal{F}_\varphi^{<\omega}$ , the set of its compact elements, is a join-semilattice with 0 and c)  $\mathcal{F}_\varphi$  is isomorphic to  $J(\mathcal{F}_\varphi^{<\omega})$ .

A basic relationship between closure operators and independent sets is this.

**Theorem 4.2.** *Let  $\varphi$  be a closure operator on a set  $E$ . The following properties are equivalent:*

- (i)  $E$  contains an independent set of size  $\kappa$ ;
  - (ii)  $\mathcal{F}_\varphi$  contains a subset isomorphic to  $\mathcal{P}(\kappa)$ ;
- If  $\varphi$  is algebraic, these two properties are equivalent to:*
- (iii)  $\mathcal{F}_\varphi^{<\omega}$  contains a subset isomorphic to  $[\kappa]^{<\omega}$ .

FIGURE 4.1.  $\Delta$ 

The proof is almost immediate and we will not give it. Via the existence of an independent set, it shows that the order containment and the semilattice containment of  $[\kappa]^{<\omega}$  are equivalent, and that the same holds for  $\mathcal{P}(\kappa)$ .

If  $P$  is a join-semilattice with 0, then  $J(P)$  is the lattice of closed sets of an algebraic closure operator on  $P$  whose independent sets are those defined above in semilattice terms. In this case, Theorem 4.1 follows immediately from Theorem 4.2.

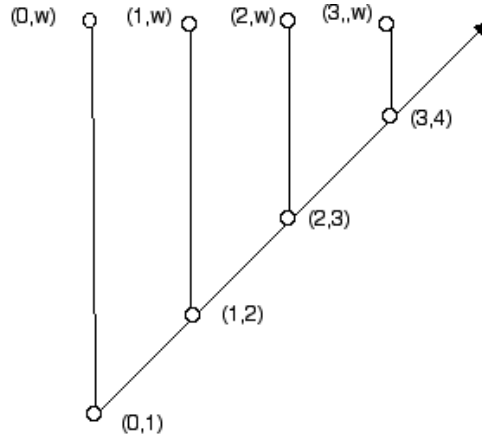
In the case of distributive lattices, the order (or join-subsemilattice) containment of  $[\omega]^{<\omega}$  can be replaced by some lattice containment. Given a poset  $P$ , let  $I_{<\omega}(P)$  be the set of finitely generated initial segments of  $P$ . Let  $\Delta := \{(i, j) : i < j \leq \omega\}$  ordered so that  $(i, j) \leq (i', j')$  if and only if either  $j \leq i'$  or  $i = i'$  and  $j \leq j'$ . Let  $\Gamma := \{(i, j) \in \Delta : j = i + 1 \text{ or } j = \omega\}$  equipped with the induced ordering. The posets  $\Delta$  and  $\Gamma$  are two well-founded meet-semilattices whose maximal elements form an infinite antichain. The posets  $I_{<\omega}(\Delta)$  and  $I_{<\omega}(\Gamma)$  are two well-founded distributive lattices containing a subset isomorphic to  $[\omega]^{<\omega}$ .

**Theorem 4.3.** *Let  $T$  be a distributive lattice. The following properties are equivalent:*

- (i)  $T$  contains a subset isomorphic to  $[\omega]^{<\omega}$ ;
- (ii) The lattice  $[\omega]^{<\omega}$  is a quotient of some sublattice of  $T$ ;
- (iii)  $T$  contains a sublattice isomorphic to  $I_{<\omega}(\Gamma)$  or to  $I_{<\omega}(\Delta)$ .

**Corollary 4.1.** *Let  $T$  be a distributive lattice. If  $T$  contains a subset isomorphic to  $[\omega]^{<\omega}$  then it contains a well-founded sublattice  $T'$  with the same property.*

One of the main ingredient in our proof of Theorem 4.3 is Ramsey's theorem [37] applied as in [11]. The posets  $\Delta$  (with a top element added) and  $\Gamma$  have been considered previously in [23] and [24]. The poset  $\delta$  obtained from  $\Delta$  by leaving out

FIGURE 4.2.  $\Gamma$ 

the maximal elements was also considered by E. Corominas in 1970 as a variant of an example built by R. Rado [36]. The results presented here are contained in part in Chapter 1 of the doctoral thesis of the first author presented before the University Claude-Bernard (Lyon1) december 18th, 1992 [6], and announced in [7].

## 2. Initial segments, ideals and a proof of Theorem 4.1

Our definitions and notations are standard and agree with [15] except on minor points that we will mention. Let  $P$  and  $Q$  be two posets. A map  $f : P \rightarrow Q$  is *order-preserving* if  $x \leq y$  in  $P$  implies  $f(x) \leq f(y)$  in  $Q$ ; this is an *embedding* if the converse also holds; this is an *order-isomorphism* if in addition  $f$  is onto. We say that  $P$  and  $Q$  are *isomorphic*, or have the same *order-type*, in notation  $P \cong Q$ , if there is an order-isomorphism from  $P$  onto  $Q$ . We also say that  $P$  *embeds* into  $Q$  if there is an embedding from  $P$  into  $Q$ , a fact we denote  $P \leq Q$ . We denote  $\omega$  the order-type of  $\mathbb{N}$ , the set of natural integers,  $\omega^*$  the order-type of the set of negative integers. Let  $P$  be a join-semilattice with a 0, an element  $x \in P$  is *join-irreducible* if it is distinct from 0, and if  $x = a \vee b$  implies  $x = a$  or  $x = b$  (this is slight difference with [15]).

If  $P$  is a poset, a subset  $I$  of  $P$  is an *initial segment* of  $P$  if  $x \in P$ ,  $y \in I$  and  $x \leq y$  imply  $x \in I$ . If  $A$  is a subset of  $P$ , then  $\downarrow A := \{x \in P : x \leq y \text{ for some } y \in A\}$  denotes the least initial segment containing  $A$ . If  $I := \downarrow A$  we say that  $I$  is *generated* by  $A$  or  $A$  is *cofinal* in  $I$ . If  $A := \{a\}$  then  $I$  is a *principal initial segment* and we write  $\downarrow a$  instead of  $\downarrow \{a\}$ . The poset  $P$  is  $\downarrow$ -closed, if the intersection of two principal initial segments of  $P$  is a finite union, possibly empty, of principal initial segments. A *final segment* of  $P$  is any initial segment of  $P^*$ , the *dual* of  $P$ . We denote by  $\uparrow A$  the final segment generated by  $A$ . If  $A := \{a\}$  we write  $\uparrow a$  instead of  $\uparrow \{a\}$ . The poset  $P$  is  $\uparrow$ -closed if its dual  $P^*$  is  $\downarrow$ -closed. A subset  $I$  of  $P$  is *up-directed* if every pair of elements of  $I$  has a common upper-bound in  $I$ . An *ideal* is a non-empty up-directed initial segment of  $P$  (in some other texts, the empty set is an ideal). We denote  $I(P)$ , resp.  $I_{<\omega}(P)$ , resp.  $J(P)$ , the set of initial segments,

resp. finitely generated initial segments, resp. ideals of  $P$  ordered by inclusion and we set  $J_*(P) := J(P) \cup \{\emptyset\}$ ,  $I_0(P) := I_{<\omega}(P) \setminus \{\emptyset\}$ . We note that  $I_{<\omega}(P)$  is the set of compact elements of  $I(P)$ , in particular  $J(I_{<\omega}(P)) \cong I(P)$ . Moreover  $I_{<\omega}(P)$  is a lattice, and in fact a distributive lattice, if and only if  $P$  is  $\downarrow$ -closed. We also note that  $J(P)$  is the set of join-irreducible elements of  $I(P)$ ; moreover,  $I_{<\omega}(J(P)) \cong I(P)$  whenever  $P$  has no infinite antichain.

**Lemma 4.1.** *Let  $P, Q$  be two posets. a) If  $P \leq Q$  then  $J(P) \leq J(Q)$ ; the converse holds if  $Q$  is a chain. b) If  $P$  and  $Q$  are join-semilattices (resp. meet-semilattices), and if  $P$  embeds into  $Q$  by a join-preserving (resp. meet-preserving) map then there is an embedding from  $J(P)$  into  $J(Q)$  by a map preserving arbitrary joins (resp. arbitrary meets).*

**Proof.** Let  $f$  be an order-preserving map from  $P$  into  $Q$ . The map  $\psi_f : J(P) \rightarrow J(Q)$  defined by  $\psi_f(I) := \downarrow f[I]$  (where  $f[I] := \{f(x) : x \in I\}$ ) preserves suprema of up-directed subsets. This is an embedding provided that  $f$  is an embedding. Conversely, suppose  $J(P) \leq J(Q)$ ; if  $Q$  is a chain, then  $I(P) \cong 1 + J(P) \leq 1 + J(Q) \cong I(Q)$ ; let  $f$  be an embedding from  $I(P)$  into  $I(Q)$ . For  $x \in P$ , choose  $g(x)$  in  $f(\downarrow x) \setminus f(\{y : y < x\})$ . The map  $g$  is an embedding from  $P$  into  $Q$ . This proves a). It is easy to check that the map  $\psi_f$  satisfies the properties stated in b).  $\square$

**Proposition 4.1.** *Let  $K$  be a poset such that for every  $x \in K$  the initial segment  $\downarrow x$  is finite and the initial segment  $K \setminus \uparrow x$  is a finite union of ideals. For every poset  $P$ , the following properties are equivalent:*

- (i)  $K \leq P$ ;
- (ii)  $J(K) \leq J(P)$ .

**Proof.** (i)  $\Rightarrow$  (ii) Lemma 4.1.

(ii)  $\Rightarrow$  (i) Let  $f$  be a map from  $J(K)$  into  $J(P)$ . For  $x \in K$ , put  $F(x) := \{f(J) : x \notin J \in J(K)\}$  and  $R(x) := \bigcup F(x)$  and  $C(x) := f(\downarrow x) \setminus R(x)$ .

**Claim**  $C(x)$  is non-empty.

Indeed, if  $x$  is the least element of  $K$  then  $R(x)$  is empty, hence  $C(x) = f(\downarrow x)$  which is non-empty. If  $x$  is not the least element of  $K$  then  $K \setminus \uparrow x$  is non-empty, hence, from our hypothesis on  $K$ , is the union  $J_1 \cup J_2 \cup \dots \cup J_{n_x}$  of a non-empty family of ideals. Since the ideals of  $P$  are the join-irreducible members of the distributive lattice  $I(P)$ , it follows first that  $R(x) = f(J_1) \cup f(J_2) \cup \dots \cup f(J_{n_x})$  (indeed, if  $f(J) \subseteq R(x)$  then  $J \subseteq J_{n_i}$  for some  $i$ ). With this, it follows next that  $C(x)$  is non-empty (otherwise, from  $f(\downarrow x) \subseteq R(x)$  we would have  $f(\downarrow x) \subseteq f(J_{n_i})$ , hence  $x \in J_{n_i}$ , for some  $i$ , which is impossible) and our claim is proved.

Let  $g : K \rightarrow P$  be such that  $g(x) \in C(x)$  for every  $x \in K$ . Clearly  $x' \not\leq x''$  implies  $g(x') \not\leq g(x'')$ . Indeed, if  $x' \not\leq x''$  then  $x' \notin \downarrow x''$ , hence  $C(x') \cap f(\downarrow x'') = \emptyset$ . This, added to the fact that  $g(x') \in C(x')$ ,  $g(x'') \in f(\downarrow x'')$  and  $f(\downarrow x'')$  is an initial segment, gives  $g(x') \not\leq g(x'')$ . Hence, if such a  $g$  is order-preserving then this is an embedding. We define such an order-preserving map  $g$  by induction on the size of

$\downarrow x$ . If  $|\downarrow x| = 1$  then we choose for  $g(x)$  any element of  $C(x)$ . If  $|\downarrow x| > 1$  and if  $g(y)$  is defined for all  $y, y < x$ , then we have  $g(y) \in f(\downarrow y) \subseteq f(\downarrow x)$ , hence  $g(y) \in f(\downarrow x)$ . The set  $\{g(y) : y < x\}$  is finite, hence, it has an upper-bound  $z$  in  $f(\downarrow x)$ . Select  $g(x)$  in  $C(x) \cap \uparrow z$ .  $\square$

Hypotheses of this proposition are satisfied if  $K := [\kappa]^{<\omega}$ . Since in this case  $J(K) \cong \mathcal{P}(\kappa)$ , we obtain Theorem 4.1.

These hypotheses are also satisfied by many other posets. This is the case of join-semilattices which embed into some  $[\kappa]^{<\omega}$  as join-subsemilattices. It is not difficult to see that a join-semilattice  $K$  has this property if and only if for every  $x \in K$ , the set  $J(P)_{\neg x}^\Delta$  of completely meet-irreducible members of  $J(P)$  which do not contain  $x$  is finite (see [9]). An interesting example is  $\delta$  (note that the map  $\varphi$  from  $\delta$  into  $[\omega]^{<\omega}$  defined by  $\varphi(i, j) := \{0, \dots, j-1\} \setminus \{i\}$  is join-preserving). Posets of the form  $K := I_{<\omega}(Q)$ , where  $Q$  embeds into  $[\kappa]^{<\omega}$ , also embed into  $[\kappa]^{<\omega}$  as join-semilattices. Evidently, it is only needed to consider those not embedding  $[\kappa]^{<\omega}$  as a join-semilattice, this amounting to the fact that  $Q$  contains no infinite antichain (cf. Section 3, Corollary 4.2). A simple minded example is  $K := \omega \times \dots \times \omega$ , the direct product of finitely many copies of  $\omega$  (obtained with  $Q := \omega \oplus \dots \oplus \omega$ , the direct sum of the same number of copies of  $\omega$ ).

The sierpinskization technique leads to examples of join-semilattices satisfying the hypotheses of Proposition 4.1, but which do not embed into  $[\omega]^{<\omega}$  as join-subsemilattices. A *sierpinskization* of a countable order-type  $\alpha$  with  $\omega$  is a poset  $S_\alpha$  made of a set  $E$  and an order  $\mathcal{E}$  on  $E$  which is the intersection of two linear orders  $\mathcal{A}$  and  $\mathcal{N}$  on  $E$  such that the chains  $A := (E, \mathcal{A})$  and  $N := (E, \mathcal{N})$  have types  $\alpha$  and  $\omega$  respectively. Clearly, every principal initial segment of  $S_\alpha$  is finite. More importantly, non-principal ideals of  $S_\alpha$  form a chain: in fact, *a subset  $I$  of  $E$  is a non-principal ideal of  $S_\alpha$  if and only if  $I$  is a non-principal ideal of the chain  $A$* . Also, *if  $K$  is a sierpinskization of  $\alpha$  and  $\omega$  then for every  $x \in K$ , the initial segment  $K \setminus \uparrow x$  is a finite union of ideals if and only if no interval of  $\alpha$  has order-type  $\omega^*$*  (cf [9]).

From this, Proposition 4.1 applies to any sierpinskisation of  $\omega\alpha$  with  $\omega$ . As shown in [34], those given by a bijective map  $\psi : \omega\alpha \rightarrow \omega$  which is order-preserving on each component  $\omega \cdot \{i\}$  of  $\omega\alpha$  are all embeddable in each other, and for this reason denoted by the same symbol  $\Omega(\alpha)$ ; moreover,  $\Omega(\alpha)$  embeds into  $\Omega(\beta)$  if and only if  $\alpha$  embeds into  $\beta$ . Consequently, the sierpinskization technique allows to construct as many countable posets  $K$  satisfying the hypotheses of Proposition 4.1 as there are countable order-types. A bit more is true. *Among the representatives of  $\Omega(\alpha)$ , some are semilattices* (and among them, subsemilattices of the direct product  $\omega \times \alpha$ ). *Except for  $\alpha = \omega$ , the representatives of  $\Omega(\alpha)$  which are join-semilattices never embed into  $[\omega]^{<\omega}$  as join-semilattices* (whereas they embed as posets) [9].

The posets  $\Omega(\alpha)$  and  $I_{<\omega}(\Omega(\alpha))$  do not embed in each other as join-semilattices. They provide two examples of a join-semilattice  $P$  such that  $P$  contains no chain of type  $\alpha$  and  $J(P)$  contains a chain of type  $J(\alpha)$ . However, note that if  $\alpha$  embeds  $\omega^*$

then  $I_{<\omega}(\Omega(\alpha))$  reduces to  $[\omega]^{<\omega}$  (they embed in each other as join-semilattices). Are they substantially different examples? (see [9] for more).

### 3. Distributive lattices containing an infinite independent set: proof of Theorem 4.3

A poset  $P$  is *well-founded* if every non-empty subset has a minimal element; it is *well-quasi-ordered* (w.q.o. in brief) if it is well-founded with no infinite antichain. Let us recall the Higman's characterization of w.q.o. sets [17].

**Theorem 4.4.** *Let  $P$  be a poset. The following properties are equivalent:*

- (i)  $P$  is well-quasi-ordered;
- (ii) For every infinite sequence  $(x_n)_{n<\omega}$  of elements of  $P$ , some infinite subsequence is non-decreasing;
- (iii) For every infinite sequence  $(x_n)_{n<\omega}$  of elements of  $P$ , there are  $n < m$  such that  $x_n \leq x_m$ ;
- (iv) Every final segment of  $P$  is finitely generated;
- (v) The set  $I(P)$  of initial segments of  $P$ , ordered by inclusion, is well-founded.

The following lemma provides an alternative version of the *minimal bad-sequence* technique invented by C. St J. A. Nash-Williams [29].

**Lemma 4.2.** *Let  $P$  be a well-founded poset. If  $P$  contains an infinite antichain then it contains some infinite antichain  $A$  such that:*

- 1) for every  $x \in P$  either  $x \geq y$  for some  $y \in A$  or  $x < y$  for all  $y \in A$  but finitely many;
- 2)  $P \setminus \uparrow A$  is w.q.o.

**Proof.** Since  $P$  is well-founded, every final segment of  $P$  is generated by its minimal elements. Let  $\mathcal{F}$  be the set of non-finitely generated final segments of  $P$ . If  $X$  is an infinite antichain of  $P$  then  $\uparrow X \in \mathcal{F}$ ; moreover  $\mathcal{F}$  is closed by union of chains. Hence  $\mathcal{F}$ , ordered by inclusion, is inductive. According to Zorn's lemma, it has some maximal element  $F$ . We claim that  $A := \text{Min}(F)$  satisfies 1) and 2). Indeed, let first  $x \in P$ . Set  $F' := F \cup \uparrow x$ . Since  $F'$  is a final segment containing  $F$  then either  $F' = F$  or  $F'$  is finitely generated. In the first case  $x \geq y$  for some  $y \in A$  whereas in the latter case  $x < y$  for all  $y \in A$  but finitely many, since  $\text{Min}(F') = \{x\} \cup (A \setminus \uparrow x)$ . This proves that 1) holds. Next, let  $G$  be a final segment of  $P \setminus \uparrow A$ . Then  $G \cup \uparrow A$  is a final segment of  $P$ . Since  $\uparrow A = F$ , this final segment strictly contains  $F$  if  $G$  is non-empty, hence it is finitely generated; this implies that  $G$  is finitely generated. According to (iv) of Theorem 4.4,  $P \setminus \uparrow A$  is w.q.o.  $\square$

**Lemma 4.3.** *If  $P$  is  $\uparrow$ -closed and  $A$  is an infinite antichain satisfying condition 1) of Lemma 4.2 and distinct from  $\text{Min}(P)$  then  $P \setminus \uparrow A$  is an ideal; in particular, if  $P$  is a join-semilattice then all members of  $A$  are join-irreducible.*

**Proof.** Since  $A \neq \text{Min}(P)$  then  $P \setminus \uparrow A$  is non-empty. Let  $x, y \in P \setminus \uparrow A$ . Condition 1) insures that  $A \setminus (\uparrow x \cap \uparrow y)$  is finite. Hence  $\uparrow x \cap \uparrow y$  is infinite and since  $P$  is



$\uparrow$ -closed,  $\uparrow x \cap \uparrow y = \uparrow F$  where  $F$  is some finite set. Let  $z \in F$  such that  $\uparrow z \cap A$  is infinite. Since  $z \in F$ , we have  $x, y \leq z$  and since  $\uparrow z \cap A$  is infinite  $z < a$  for some  $a \in A$ , hence  $z \in P \setminus \uparrow A$ , proving that  $P \setminus \uparrow A$  is up-directed.

If  $P$  is a join-semilattice, it is  $\uparrow$ -closed, hence  $P \setminus \uparrow A$  is an ideal. In this case  $x \vee y \in P \setminus \uparrow A$  whenever  $x, y \in P \setminus \uparrow A$ . Hence if  $a \in A$  and  $a = x \vee y$  then  $x < a$  and  $y < a$  are impossible.  $\square$

Let us recall a basic property of join-irreducibles in a distributive lattices.

**Lemma 4.4.** *Let  $T$  be a distributive lattice and  $x \in T$  with  $x$  distinct from the least element of  $T$  if any. The following properties are equivalent:*

- (i)  $x$  is join-irreducible;
- (ii) If  $a, b \in T$  and  $x \leq a \vee b$  then  $x \leq a$  or  $x \leq b$ ;
- (iii) For every integer  $k$  and every  $a_1, \dots, a_k \in T$  if  $x \leq a_1 \vee \dots \vee a_k$  then  $x \leq a_i$  for some  $i$ ,  $1 \leq i \leq k$ .

**Corollary 4.2.** *A well-founded distributive lattice  $T$  contains no subset isomorphic to  $[\omega]^{<\omega}$  if and only if it contains no infinite antichain.*

**Proof.** If  $T$  contains an infinite antichain then, since  $T$  is well-founded, Lemmas 4.2 and 4.3 apply, hence the set  $T^\vee$  of join-irreducible elements of  $T$  contains an infinite antichain. From Lemma 4.4, this antichain is in fact an independent subset of  $T$ ; according to Theorem 4.2,  $T$  contains a subset isomorphic to  $[\omega]^{<\omega}$ . The converse is obvious.  $\square$

In the sequel we describe typical well-founded meet-semilattices containing infinite antichains.

**Lemma 4.5.** *Let  $P$  be a meet-semilattice and  $f : \Delta \rightarrow P$  be a map satisfying  $f(i, j) = f(i, \omega) \wedge f(j, \omega)$  for all  $i < j < \omega$ . Then the following properties are equivalent:*

- (i)  $f$  is meet-preserving;
- (ii)  $f$  is order-preserving;
- (iii)  $f(i, j) \leq f(k, \omega)$  for all  $i < j < k < \omega$ ;
- (iv)  $f(i, j) \leq f(j, k)$  for all  $i < j < k < \omega$ ;
- (v)  $f(i, j) \leq f(i, k)$  for all  $i < j < k < \omega$ ;
- (vi)  $f(i, j) = f(i, k) \wedge f(j, k)$  for all  $i < j < k < \omega$ ;

*Moreover, if  $f$  satisfies these conditions, then  $f$  is one-to-one if and only if it satisfies conditions a)  $f(i, j) < f(j, k)$  and b)  $f(i, j) < f(i, k)$  for all  $i < j < k < \omega$ .*

**Proof.**

(i)  $\Rightarrow$  (ii) Evident.

(ii)  $\Rightarrow$  (iii) In  $\Delta$  we have  $(i, j) \leq (k, \omega)$  for all  $i < j < k < \omega$ ; if  $f$  is order-preserving, then this inequality is preserved.

(iii)  $\iff$  (iv)  $\iff$  (v)  $\iff$  (vi) Since  $f(i', j') = f(i', \omega) \wedge f(j', \omega)$  for all  $i' < j' < \omega$ , we have  $f(i, j) \wedge f(i, k) = f(i, j) \wedge f(k, \omega) = f(i, j) \wedge f(j, k) = f(i, k) \wedge f(j, k)$  for all  $i < j < k < \omega$ . The three equivalences follow. (iii)  $\Rightarrow$  (ii) Let  $x := (i, j)$  and  $x' := (i', j')$  in  $\Delta$  such that  $x < x'$ .

**Case 1**  $i = i'$  and  $j < j'$ . From the definition of  $f$  we have  $f(i, j) = f(i, \omega) \wedge f(j, \omega)$ , hence if  $j' = \omega$  then  $f(x) \leq f(x')$ . If  $j' < \omega$ , then from (v), we have also  $f(x) = f(i, j) \leq f(i, j') = f(x')$ .

**Case 2**  $j \leq i'$ . Suppose  $j' = \omega$ . If  $j < i'$  then from (iii)  $f(x) = f(i, j) \leq f(i', \omega) = f(x')$ ; if  $j = i'$  then from the definition of  $f(x)$  we have  $f(x) = f(i, j) \leq f(j, \omega) = f(x')$ . Suppose  $j' < \omega$ . From (iv) we have  $f(x) = f(i, j) \leq f(j, i') \leq f(i', j') = f(x')$  if  $j < i'$  or  $f(x) = f(i, j) \leq f(j, j') = f(i', j') = f(x')$  if  $j = i'$ .

(ii)  $\Rightarrow$  (i) We have to check that  $f(x \wedge x') = f(x) \wedge f(x')$  for all pairs  $x := (i, j)$ ,  $x' := (i', j')$  in  $\Delta$ . Since  $f$  is order-preserving, we only need to consider incomparable pairs. Let  $x := (i, j)$ ,  $x' := (i', j')$  be such a pair. We have  $i \neq i'$ . We may suppose  $i < i'$ ; in this case, since  $x$  and  $x'$  are incomparable we have  $i' < j$ , hence  $x \wedge x' = (i, i')$ .

If we suppose  $j < \omega$  and  $j' < \omega$  we have  $f(x) \wedge f(x') = f(i, j) \wedge f(i', j') = f(i, \omega) \wedge f(j, \omega) \wedge f(i', \omega) \wedge f(j', \omega) = f(i, i') \wedge f(j, \omega) \wedge f(j', \omega) = f(i, i') = f(x \wedge x')$  since  $(i, i') \leq (j, \omega) \wedge (j', \omega)$  and  $f$  is order-preserving.

If  $j < \omega$  and  $j' = \omega$  we have similarly  $f(x) \wedge f(x') = f(i, j) \wedge f(i', \omega) = f(i, \omega) \wedge f(j, \omega) \wedge f(i', \omega) = f(i, i') \wedge f(j, \omega) = f(i, i') = f(x \wedge x')$ . If  $j = \omega$  and  $j' < \omega$  we have  $f(x) \wedge f(x') = f(i, \omega) \wedge f(i', j') = f(i, \omega) \wedge f(i', \omega) \wedge f(j', \omega) = f(i, i') \wedge f(j', \omega) = f(i, i') = f(x \wedge x')$ . If  $j = j' = \omega$  we have  $f(x) \wedge f(x') = f(i, \omega) \wedge f(i', \omega) = f(i, i') = f(x \wedge x')$ .

Suppose that  $f$  satisfies conditions (i) – (vi). If  $f$  is one-to-one then from (ii) applied to  $(i, j) < (j, k)$  and  $(i, j) < (i, k)$ , we get  $f(i, j) < f(j, k)$  and  $f(i, j) < f(i, k)$  as required. Suppose that  $f$  is not one-to-one. Let  $x := (i, j)$  and  $x' := (i', j')$  be two distinct elements in  $\Delta$  such that  $f(x) = f(x')$ .

**Case 1**  $x$  and  $x'$  are comparable. We may suppose  $x < x'$ . **Subcase 1.1**  $i = i'$  and  $j < j'$ . Since  $f$  is order-preserving  $f(i, j) = f(i, j + 1)$  and condition b) is violated. **Subcase 1.2**  $j \leq i'$ . Since in this case  $(i, j) \leq (j, j + 1) \leq (i', j')$  we have  $f(i, j) = f(j, j + 1)$  and condition a) is violated.

**Case 2**  $x$  and  $x'$  are incomparable. We may suppose  $i < i' < j$ . Since  $f$  is meet-preserving, we have  $f(i, j) = f(x) = f(x') = f(x) \wedge f(x') = f(x \wedge x') = f(i, i')$ , hence  $f(i, i') = f(i, i' + 1)$  and condition b) is violated.  $\square$

Let  $V$  be the meet-semilattice made of a countable antichain and a least element added (formally  $V := \{X \subseteq \omega : |X| < 2\}$  ordered by inclusion).

**Lemma 4.6.** *Let  $P$  be a meet-semilattice. If  $P$  contains an infinite antichain  $A$  such that the set  $P_{<}(A) := \{x \in P : x < a \text{ for some } a \in A\}$  is well-quasi-ordered, then there is an infinite subset  $A'$  of  $A$  such that the meet-subsemilattice  $P'$  of  $P$  generated by  $A'$  is either isomorphic to  $\Delta$ , to  $\Gamma$  or to  $V$ .*

**Proof.** Let  $\{x_n : n < \omega\}$  be a countable subset of  $A$ . Consider the partition of  $[\mathbb{N}]^3 = \{(i, j, k) : i < j < k < \omega\}$  into the following parts:

$$R_1 := \{(i, j, k) \in [\mathbb{N}]^3 : x_i \wedge x_j \text{ incomparable to } x_i \wedge x_k\};$$

$$R_2 := \{(i, j, k) \in [\mathbb{N}]^3 : x_i \wedge x_j > x_i \wedge x_k\};$$

$$R_3 := \{(i, j, k) \in [\mathbb{N}]^3 : x_i \wedge x_j < x_i \wedge x_k\};$$

$$R_4 := \{(i, j, k) \in [\mathbb{N}]^3 : x_i \wedge x_j = x_i \wedge x_k = x_j \wedge x_k\};$$

$$R_5 := \{(i, j, k) \in [\mathbb{N}]^3 : x_i \wedge x_j = x_i \wedge x_k < x_j \wedge x_k\}.$$

From Ramsey's theorem[37] there is an infinite subset  $H$  of  $\mathbb{N}$  such that  $[H]^3 \subseteq R_i$  for some  $i \in \{1, 2, 3, 4, 5\}$ . Since  $P_<(A)$  is well-quasi-ordered,  $[H]^3$  is neither included in  $R_1$ , nor in  $R_2$ . In the remaining cases, we select an infinite subset  $H'$  of  $H$  and, setting  $A' := \{x_n : n \in H'\}$ , we define a meet-preserving and one-to-one map  $h_{H'}$  from  $\Delta$ ,  $\Gamma$  or  $V$  onto the meet-semilattice  $P'$  generated by  $A'$ . In order to do so, we denote by  $\theta_K$  the unique order-isomorphism from  $\mathbb{N}$  onto an infinite subset  $K$  of  $\mathbb{N}$ .

**Case 1**  $[H]^3 \subseteq R_3$ . Let  $h_H : \Delta \rightarrow P$  be defined by  $h_H(x) := x_{\theta_H(i)}$  if  $x := (i, \omega)$  and  $h_H(x) := h_H(i, \omega) \wedge h_H(j, \omega)$  if  $x := (i, j)$  with  $i < j < \omega$ . The map  $h_H$  satisfies condition (v) of Lemma 4.5, hence, it is meet-preserving. It also satisfies condition b) of Lemma 4.5 but it is not necessarily one-to-one. Let  $H'$  be the image of  $2\mathbb{N}$ , the set of even integers, by  $\theta_H$ , and let  $\theta_{H'}$  be the unique order-isomorphism from  $\mathbb{N}$  onto  $H'$ . Like  $h_H$ , the map  $h_{H'} : \Delta \rightarrow P$  is also meet-preserving and satisfies condition b). It also satisfies condition a). Indeed, let  $i < j < k < \omega$ . Since  $\theta_{H'}(n) = \theta_H(2n)$  for every  $n < \omega$ , we have  $h_{H'}(i, j) = h_H(2i, 2j) \leq h_H(2j, 2j+1) < h_H(2j, 2k) = h_{H'}(j, k)$ . According to Lemma 4.5,  $h_{H'}$  is one-to-one.

**Case 2**  $[H]^3 \subseteq R_4$ .

In this case we have  $x_i \wedge x_j = x_{i'} \wedge x_{j'}$  for every  $(i, j), (i', j') \in [H]^2$ ; we denote  $a$  this common value. Let  $H' := H$ ,  $h_{H'} : V \rightarrow P$  defined by setting  $h_{H'}(x) := x_{\theta_{H'}(i)}$  if  $x := \{i\}$  and  $h_{H'}(x) := a$  if  $x := \emptyset$ .

**Case 3**  $[H]^3 \subseteq R_5$ .

Set  $H' := H$ ; let  $h_{H'} : \Gamma \rightarrow P$  be the map defined by setting  $h_{H'}(x) := x_{\theta_{H'}(i)}$  if  $x := (1, i)$  and  $h_{H'}(x) := x_{\theta_{H'}(i)} \wedge x_{\theta_{H'}(j)}$ , where  $j$  is any element of  $H'$  such that  $i < j$ , if  $x := (0, i)$ .  $\square$

**Lemma 4.7.** *Let  $P$  be a well-founded meet-semilattice. The image of  $P$  by a meet-preserving map  $f$  is well-founded.*

**Proof.** Let  $Y$  be a non-empty subset of  $f(P)$ , then the image by  $f$  of every minimal element  $a$  in  $f^{-1}(Y)$  is minimal in  $Y$ . Indeed, let  $b \in Y$  with  $b \leq f(a)$ . We have  $b = f(a')$  for some  $a' \in f^{-1}(Y)$ . Since  $f$  is meet-preserving, we have  $b = f(a) \wedge f(a') = f(a \wedge a')$ . As a first consequence, we get  $a \wedge a' \in f^{-1}(Y)$  which, in turns, gives  $a = a \wedge a'$  since  $a$  is minimal in  $f^{-1}(Y)$ . Next, we get  $b = f(a)$  proving the minimality of  $f(a)$ .  $\square$

**Theorem 4.5.** *Let  $P$  be a meet-semilattice. The following properties are equivalent:*

- (i) *The meet-subsemilattice generated by some infinite antichain of  $P$  is well-founded;*
- (ii) *The meet-subsemilattice  $Q$  generated by some infinite antichain  $A$  of  $P$  is such that  $\{x \in Q : x < y \text{ for some } y \in A\}$  is well-quasi-ordered;*
- (iii)  *$P$  contains a meet-subsemilattice isomorphic to  $\Gamma$  or to  $\Delta$  or to  $V$ ;*
- (iv) *There is a meet-preserving map  $f : \Delta \rightarrow P$  whose image contains an infinite antichain.*

**Proof.** (i)  $\Rightarrow$  (ii) Lemma 4.2.

(ii)  $\Rightarrow$  (iii) Lemma 4.6.

(iii)  $\Rightarrow$  (iv) Each of the meet-semilattices  $\Gamma$  and  $V$  is a quotient of  $\Delta$  by a meet-preserving map. Indeed, define  $g : \Delta \rightarrow \Gamma$  by  $g(i, j) := (1, i)$  if  $j = \omega$  and  $g(i, j) := (0, i)$  otherwise. Clearly  $g(i, \omega) \wedge g(j, \omega) = (1, i) \wedge (1, j) = (0, i) = g(i, j)$  for all  $i < j < \omega$ ; hence, from Lemma 4.5,  $g$  is meet-preserving. Similarly, define  $h : \Delta \rightarrow V$  by  $h(i, j) := \{i\}$  if  $j = \omega$  and  $h(i, j) := \emptyset$  otherwise; the map  $h$  is meet-preserving too.

(iv)  $\Rightarrow$  (i) Lemma 4.7. □

**Lemma 4.8.** *Let  $T$  be a distributive lattice containing an independent set  $L$  with at least two elements. Let  $\langle L \rangle$  be the sublattice of  $T$  generated by  $L$ . Then:*

(i) *every  $a \in L$  is join-irreducible in  $\langle L \rangle$ ;*

(ii) *the map  $\varphi$  from  $\langle L \rangle$  into  $\mathcal{P}(L)$  defined by  $\varphi(x) := \{a \in L : a \leq x\}$  is a lattice-homomorphism whose image is  $[L]^{<\omega}$ .*

**Proof.** (i) Let  $a \in L$ . Since  $|L| \geq 2$ ,  $a$  is distinct from the least element of  $\langle L \rangle$  (if any). Let  $u, v \in \langle L \rangle$  such that  $a \not\leq u$  and  $a \not\leq v$ . Since  $\langle L \rangle$  is distributive, we can write  $u = u_1 \wedge u_2 \wedge \cdots \wedge u_p$  and  $v = v_1 \wedge v_2 \wedge \cdots \wedge v_q$ , where every  $u_i$  is the supremum of a finite subset  $U_i$  of  $L$  and every  $v_j$  is the supremum of a finite subset  $V_j$  of  $L$ . Since  $a \not\leq u$ , there is some  $u_i$  such that  $a \not\leq u_i$  and similarly, since  $a \not\leq v$ , there is some  $v_j$  such that  $a \not\leq v_j$ . Since  $u_i = \bigvee U_i$  then  $a \notin U_i$  and, since  $v_j = \bigvee V_j$ , then  $a \notin V_j$ . So  $a \notin U_i \cup V_j$ . Since  $L$  is independent, it follows that  $a \not\leq \bigvee (U_i \cup V_j) = u_i \vee v_j$ . Since  $u \vee v \leq u_i \vee v_j$  then  $a \not\leq u \vee v$  proving that  $a$  is join-irreducible.

(ii) As a map from  $\langle L \rangle$  into  $\mathcal{P}(L)$ ,  $\varphi$  is a lattice-homomorphism. Indeed, let  $x, y \in \langle L \rangle$ . The equality  $\varphi(x \wedge y) = \varphi(x) \cap \varphi(y)$  is clear. Let us check that the equality  $\varphi(x \vee y) = \varphi(x) \cup \varphi(y)$  holds. Obviously  $\varphi(x) \cup \varphi(y) \subseteq \varphi(x \vee y)$ . For the reverse inclusion, let  $a \in \varphi(x \vee y)$ . From (i)  $a$  is join-irreducible in  $\langle L \rangle$ , hence  $a \leq x$  or  $a \leq y$  that is  $x \in \varphi(x) \cup \varphi(y)$ . Since  $|L| \geq 2$ ,  $[L]^{<\omega}$  is the sublattice of  $\mathcal{P}(L)$  generated by  $L' := \{\{a\} : a \in L\}$ . Since  $\varphi$  is a lattice-homomorphism and  $L'$  is the image of  $L$  it follows that  $[L]^{<\omega}$  is the image of  $\langle L \rangle$ . □

**Lemma 4.9.** *Let  $T$  be a lattice and  $\varphi$  be a lattice-homomorphism from  $T$  onto  $[\omega]^{<\omega}$ . Then there is a meet-preserving map  $f$  from  $\Delta$  into  $T$  such that  $\varphi \circ f(i, \omega) = \{i\}$  for every  $i < \omega$ .*

**Proof.** First, we define  $f(i, \omega)$  for  $i < \omega$ . Denote  $B_0 := \{x \in T : \varphi(x) = \emptyset\}$  and  $A_i := \{x \in T : \varphi(x) = \{i\}\}$  for  $i < \omega$ . Since  $\varphi$  is onto, these sets are non-empty. Let  $b_0 \in B_0, a_0 \in A_0$  and  $a_1 \in A_1$ . Set  $f(0, \omega) := a_0 \vee b_0$  and  $f(1, \omega) := a_1 \vee b_0$ . Since  $\varphi$  is a lattice-homomorphism we have  $\varphi(f(i, \omega)) = \varphi(a_i) \vee \varphi(b_0) = \{i\} \cup \emptyset = \{i\}$  for  $i = 0, 1$ . Let  $k \geq 2$ . Suppose  $f(i, \omega)$  defined for  $i < k$ . Choose  $a_k$  in  $A_k$  and set  $b_k := \bigvee \{f(i, \omega) \wedge f(j, \omega) : i < j < k\}$ . Put  $f(k, \omega) := b_k \vee a_k$ . We have  $\varphi(b_k) = \emptyset$ , and so  $\varphi(f(k, \omega)) = \varphi(b_k) \vee \varphi(a_k) = \{k\}$ . Finally, put  $f(i, j) := f(i, \omega) \wedge f(j, \omega)$  for all  $i < j < \omega$ .

The map  $f$  satisfies condition (iii) of Lemma 4.5. Indeed, let  $i < j < k < \omega$ , we have  $f(i, j) := f(i, \omega) \wedge f(j, \omega) \leq b_k \leq b_k \vee a_k =: f(k, \omega)$ , hence  $f(i, j) \leq f(k, \omega)$ ,

as required. Hence, according to Lemma 4.5,  $f$  is meet-preserving.

□

**Lemma 4.10.** *Let  $f : P \rightarrow T$  be a meet-preserving map from a meet-semilattice  $P$  into a distributive lattice  $T$ .*

*(i) The map  $f^\vee$  from  $I_0(P) := I_{<\omega}(P) \setminus \{\emptyset\}$  into  $T$ , defined by  $f^\vee(A) := \bigvee \{f(a) : a \in A\}$ , is a lattice-homomorphism.*

*(ii)  $f^\vee$  is injective if and only if 1)  $f$  is injective and 2) for every  $x \in P$ , every finite non-empty subset  $X$  of  $P$ , the equality  $f(x) = \bigvee f(X)$  implies  $x \in X$ .*

**Proof.** (i) Let  $A, B \in I_0(P)$ . We have  $f^\vee(A \cup B) = \bigvee \{f(c) : c \in A \cup B\} = (\bigvee \{f(a) : a \in A\}) \vee (\bigvee \{f(b) : b \in B\}) = f^\vee(A) \vee f^\vee(B)$ , hence  $f^\vee$  is join-preserving. By definition  $f^\vee(A) \wedge f^\vee(B) = (\bigvee \{f(a) : a \in A\}) \wedge (\bigvee \{f(b) : b \in B\})$ . Since  $A$  and  $B$  are finitely generated,  $T$  is distributive and  $f$  is meet-preserving, we have  $f^\vee(A) \wedge f^\vee(B) = \bigvee \{f(a) \wedge f(b) : a \in A, b \in B\} = \bigvee \{f(a \wedge b) : a \in A, b \in B\}$ . Since  $A$  and  $B$  are initial segments of  $P$  then  $A \cap B = \downarrow \{a \wedge b : a \in A, b \in B\}$ . Hence,  $f^\vee(A) \wedge f^\vee(B) = \bigvee \{f(c) : c \in A \cap B\} = f^\vee(A \cap B)$ , proving that  $f^\vee$  is meet-preserving.

(ii) Let  $T' := I_0(P)$  and let  $i : P \rightarrow T'$  be the map defined by  $i(x) = \downarrow x$ . This map is injective and maps  $P' := P \setminus \text{Min}(P)$  on  $T'^\vee$ , the set of join-irreducible elements of  $T'$ . Clearly  $f = f^\vee \circ i$ . Hence, if  $f^\vee$  is injective, then  $f$  too and Condition 1) is satisfied. Moreover, the image by  $f^\vee$  of  $T'^\vee$  is the set of join-irreducible elements of the sublattice  $f^\vee(T')$ . Hence, the image of  $P'$  by  $f$  is the set of join-irreducible members of the sublattice  $f^\vee(T')$ . This amounts to Condition 2). Conversely, if these two conditions are satisfied, then  $f(P')$  is equal to the set of join-irreducible elements of  $f^\vee(T')$  and this insures that  $f^\vee$  is a lattice-isomorphism from  $T'$  onto  $f^\vee(T')$ . □

**Corollary 4.3.** *If a meet-semilattice  $P$  embeds into a meet-semilattice  $Q$  as a subsemilattice then  $I_0(P)$  embeds into  $I_0(Q)$  as a sublattice.*

**Proof.** Let  $g : P \rightarrow Q$  be a one-to-one meet-preserving map. Let  $T := I_0(Q)$  and  $f : P \rightarrow T$  defined by  $f(x) := \downarrow g(x)$ . Conditions 1) and 2) in (ii) of Lemma 4.10 are satisfied, hence  $f^\vee : I_0(P) \rightarrow T$  is a one-to-one lattice-homomorphism. □

**Lemma 4.11.** *Let  $T$  be a distributive lattice. If the meet-subsemilattice of  $T$  generated by a subset  $A$  of  $T$  is isomorphic to  $\Delta$ , to  $\Gamma$  or to  $V$ , then the sublattice of  $T$  generated by some infinite subset  $A'$  of  $A$  is either isomorphic to  $I_0(\Delta)$ , to  $I_0(\Gamma)$  or to  $[\omega]^{<\omega}$ .*

**Proof.** Let  $P$  one of the meet-semilattices  $\Delta$ ,  $\Gamma$  or  $V$ , and  $f : P \rightarrow T$  be a one-to-one meet-preserving map. According to (i) of Lemma 4.10,  $f$  extends to a lattice-homomorphism  $f^\vee : I_0(P) \rightarrow T$ . According to (ii) of Lemma 4.10,  $f^\vee$  is one-to-one if for every  $x \in P$ , every finite non-empty subset  $X$  of  $P$ , the equality  $f(x) = \bigvee f(X)$  implies  $x \in X$ . This condition is satisfied if  $P$  is  $\Gamma$  or  $V$  (indeed in this case, every  $x \in P \setminus \text{Min}(P)$  is completely join-irreducible, that is  $\{y \in P : y < x\}$  has a largest element). This is not necessarily the case if  $P := \Delta$ . In this case, set  $g : \Delta \rightarrow \Delta$  defined by setting  $g(i, j) := (2i, \omega)$  if  $j = \omega$  and  $g(i, j) := (2i, 2j)$  if

$j < \omega$  and  $h := f \circ g$ . Clearly  $g$  is a one-to-one meet-preserving map, hence  $h$  is a one-to-one meet-preserving map from  $\Delta$  into  $T$ . Moreover  $h$  satisfies condition 2) of (ii) of Lemma 4.10. Indeed, let  $x := (i, j) \in \Delta$  and  $X$  be a finite subset of  $\Delta$  such that  $h(x) = \bigvee h(X)$ . Since  $h$  is an embedding, this entail that  $x = \bigvee X$  in  $\Delta$ . If  $x \notin X$  then  $i = \text{Max}\{j' : (i', j') \in X\}$  and  $j = i + 1$ . Since  $f$  is an embedding, we have  $h(i', j') = f(2i', 2j') \leq f(2i, 2i + 1) < f(2i, 2j) = h(x)$  for all  $(i', j') \in X$ , contradicting  $h(x) = \bigvee h(X)$ . This insures that  $h^\vee$  is a one-to-one homomorphism from  $I_0(\Delta)$  into  $T$ , the image being generated by the subset  $A' := \{h(i, \omega) : i < \omega\}$ .

□

**Lemma 4.12.** *If  $P$  is  $\Gamma$  or  $\Delta$ , the lattices  $I_0(P)$  and  $I_{<\omega}(P)$  embed in each other as sublattices. On an other hand, the lattices  $I_{<\omega}(\Gamma)$  and  $I_{<\omega}(\Delta)$  do not embed in each other as sublattices.*

**Proof.** If  $P$  is  $\Gamma$  or  $\Delta$  then  $1 + P$  embeds into  $P$  by a meet-preserving map. From Corollary 4.3,  $I_0(1 + P)$  embeds into  $I_0(P)$  as a sublattice. Since  $I_{<\omega}(P)$  is isomorphic to  $I_0(1 + P)$  the desired conclusion follows.

As a poset  $I_{<\omega}(\Delta)$  does not embed into  $I_{<\omega}(\Gamma)$ . Indeed, the chain  $\omega + 1$  embeds into  $\Delta$  hence into  $I_{<\omega}(\Delta)$  whereas it does not embed into  $I_{<\omega}(\Gamma)$  since this poset embeds into  $[\omega]^{<\omega}$ , a poset which does not embed  $\omega + 1$ . On an other hand,  $I_{<\omega}(\Gamma)$  embeds into  $I_{<\omega}(\Delta)$  as a poset, but not as a sublattice. Indeed, in  $\Gamma$  the antichain  $\{(i, \omega) : i < \omega\}$  is such that  $(i, \omega) \wedge (j, \omega) = (i, \omega) \wedge (k, \omega)$  for  $i < j < k < \omega$ . Consequently, if  $I_{<\omega}(\Gamma)$  was embeddable into  $I_{<\omega}(\Delta)$  as a sublattice then  $I_{<\omega}(\Delta)$  would contain an antichain  $\{U_n : n < \omega\}$  such that  $U_i \cap U_j = U_i \cap U_k$  for all  $i < j < k$ . But this equality does not hold even for  $i = 0, j = 1$  and all  $k$ . Otherwise, if  $u \in U_0 \setminus U_1$ , then  $u \in U_0 \setminus U_k$ , for all  $k, k > 1$ . Hence,  $I_{<\omega}(\Delta \setminus \uparrow u)$  contains the infinite antichain  $\{U_k : 1 < k\}$  from which it follows that  $u = (i, \omega)$  for some  $i$  (indeed, if  $u := (i, j)$  with  $i < j < \omega$  then, since  $\Delta \setminus \uparrow u$  is covered by finitely many chains,  $I_{<\omega}(\Delta \setminus \uparrow u)$  contains no infinite antichain). Let  $v := (i', j') < u$ ; then  $v \in U_0$  and, since  $j' < \omega$ , the previous argument gives  $v \in U_1$ . Since  $U_1$  is finitely generated, it contains some element  $u_1$  above infinitely many elements below  $u$ . The structure of  $\Delta$  imposes  $u_1 = u$  which is impossible. □

**3.1. Proof of Theorem 4.3.** (i)  $\Rightarrow$  (ii) Let  $L$  be an independent subset of  $T$ . From Lemma 4.8, the sublattice  $\langle L \rangle$  generated by  $L$  has  $[L]^{<\omega}$  as a quotient. To conclude, choose  $L$  to be countable.

(ii)  $\Rightarrow$  (iii) If  $[\omega]^{<\omega}$  is a quotient of some sublattice of  $T$  then, according to Lemma 4.9 there is a meet-preserving map from  $\Delta$  into  $T$  whose image contains an infinite antichain. According to (iii) of Theorem 4.5,  $T$  contains either  $\Delta$ , or  $\Gamma$  or  $V$  as a meet-subsemilattice, hence, from Lemma 4.11 and Lemma 4.12,  $T$  contains either  $I_{<\omega}(\Delta)$ ,  $I_{<\omega}(\Gamma)$  or  $[\omega]^{<\omega}$  as a sublattice. Since  $[\omega]^{<\omega}$  has a sublattice isomorphic to  $I_{<\omega}(\Gamma)$  the conclusion follows.

(iii)  $\Rightarrow$  (i)  $I_{<\omega}(\Gamma)$  and  $I_{<\omega}(\Delta)$  contain an infinite independent set (namely  $\{\downarrow (i, \omega) : i < \omega\}$ ). □

## Notations, basic definitions and facts

Poset, qoset, chain:

If  $(P, \leq)$  is a partially ordered set, shortly a *poset*, we will often just write  $P$  for  $(P, \leq)$ . We write  $x \leq y$  for  $(x, y) \in \leq$ . A *qoset* is a quasi-ordered set and a linearly ordered poset is a *chain*.

Initial segment, principal,  $I(P)$ ,  $I_{<\omega}(P)$ ,  $I_0(P)$ ,  $\downarrow A$ :

A subset  $I$  of  $P$  is an *initial segment* if  $x \leq y$  and  $y \in I$  imply  $x \in I$ . We denote by  $I(P)$  the set of initial segments of  $P$  ordered by inclusion. Let  $A$  be a subset of  $P$ , then:  $\downarrow A := \{y \in P : y \leq x \text{ for some } x \in A\}$ . If  $A$  contains only one element  $a$ , we write  $\downarrow a$  instead of  $\downarrow \{a\}$ . An initial segment generated by a singleton is *principal* and it is *finitely generated* if it is generated by a finite subset of  $P$ . We denote by  $I_{<\omega}(P)$  the set of finitely generated initial segments and  $I_0(P) := I_{<\omega}(P) \setminus \{\emptyset\}$ .

Up-directed, ideal,  $J(P)$ ,  $\downarrow$ -closed:

A subset  $I$  of  $P$  is *up-directed* if every pair of elements of  $I$  has a common upper-bound in  $I$ . An *ideal* is a non-empty up-directed initial segment of  $P$ . We denote  $J(P)$ , the set of ideals of  $P$  ordered by inclusion and we set  $J_*(P) := J(P) \cup \{\emptyset\}$ . The poset  $P$  is  $\downarrow$ -*closed* if the intersection of two principal initial segments of  $P$  is a finite union, possibly empty, of principal initial segments.

Dual, final segment:

The *dual* of  $P$  is the poset obtained from  $P$  by reversing the order; we denote it by  $P^*$ . A subset which is an initial segment of  $P^*$  will be called a final segment of  $P$ .

Order-preserving, embedding, order-isomorphism:

Let  $P$  and  $Q$  be two posets. A map  $f : P \rightarrow Q$  is *order-preserving* if  $x \leq y$  in  $P$  implies  $f(x) \leq f(y)$  in  $Q$ ; this is an *embedding* if  $x \leq y$  in  $P$  is equivalent to  $f(x) \leq f(y)$  in  $Q$ ; if, in addition,  $f$  is onto, then this is an *order-isomorphism*.

Equimorphic posets, order type:

We say that  $P$  *embeds* into  $Q$  if there is an embedding from  $P$  into  $Q$ , a fact we denote  $P \leq Q$ ; if  $P \leq Q$  and  $Q \leq P$  then  $P$  and  $Q$  are *equimorphic*, we denote  $P \equiv Q$ . If there is an order-isomorphism from  $P$  onto  $Q$  we say that  $P$  and  $Q$  are *isomorphic* or have the same *order type*, a fact we denote  $P \cong Q$ .

$\omega, \omega^*, \eta$ :

We denote  $\omega$  the order type of  $\mathbb{N}$ , the set of natural integers,  $\omega^*$  the order type of the set of negative integers and  $\eta$  the order type of  $\mathbb{Q}$ , the set of rational numbers.

Well-founded, well-quasi-ordered, well-ordered, ordinal, scattered:

A poset  $P$  is *well-founded* if the order type  $\omega^*$  does not embed into  $P$ . If furthermore,  $P$  has no infinite antichain then  $P$  is *well-quasi-ordered* (wqo). A well-founded chain is *well-ordered*; its order type is an *ordinal*. A poset  $P$  is *scattered*

if it does not contain a copy of  $\eta$ , the chain of rational numbers.

$\mathfrak{P}(E), [E]^{<\omega}, \mathcal{F}^{<\omega}, \mathcal{F}^\cup$ :

Let  $E$  be a set, we denote  $[E]^{<\omega}$  (resp.  $\mathfrak{P}(E)$ ), the set, ordered by inclusion, consisting of finite (resp. arbitrary) subsets of  $E$ . If  $\mathcal{F}$  is a subset of  $\mathfrak{P}(E)$ , we denote  $\mathcal{F}^{<\omega}$  (resp.  $\mathcal{F}^\cup$ ) the collection of finite (resp. arbitrary) unions of members of  $\mathcal{F}$  ordered by inclusion.

Lexicographic sum, ordinal sum, lexicographic product:

If  $(P_i)_{i \in I}$  is a family of posets indexed by a poset  $I$ , the *lexicographic sum* of this family is the poset, denoted  $\Sigma_{i \in I} P_i$ , defined on the disjoint union of the  $P_i$ , that is formally the set of  $(i, x)$  such that  $i \in I$  and  $x \in P_i$ , equipped with the order  $(i, x) \leq (j, y)$  if either  $i < j$  in  $I$  or  $i = j$  and  $x \leq y$  in  $P_i$ . When  $I$  is the finite chain  $n := \{0, 1, \dots, n-1\}$  this sum is denoted  $P_0 + P_1 + \dots + P_{n-1}$ . When  $I := \omega$  this sum is denoted  $\Sigma_{i < \omega} P_i$  or  $P_0 + P_1 + \dots + P_n + \dots$ . We denote  $\Sigma_{i \in I}^* P_i$  for  $\Sigma_{i \in I} P_i$ ; when  $I := \omega$  we denote  $\Sigma_{i < \omega}^* P_i$  or  $\dots + P_n + \dots + P_1 + P_0$ . When  $I$  (resp.  $I^*$ ) is well ordered, or is an ordinal, we call it *ordinal sum* (resp. *antiordinal sum*) instead of lexicographic sum. When all the  $P_i$  are equal to the same poset  $P$ , the lexicographic sum is denoted  $P.I$ , and called the *lexicographic product* of  $P$  and  $I$ .

Direct product, direct sum:

The *direct product* of  $P$  and  $Q$  denoted  $P \times Q$  is the set of  $(p, q)$  for  $p \in P$  and  $q \in Q$ , equipped with the product order; that is  $(p, q) \leq (p', q')$  if  $p \leq p'$  and  $q \leq q'$ . The *direct sum* of  $P$  and  $Q$  denoted  $P \oplus Q$  is the disjoint union of  $P$  and  $Q$  with no comparability between the elements of  $P$  and the elements of  $Q$  (formally  $P \oplus Q$  is the set of couples  $(x, 0)$  with  $x \in P$  and  $(y, 1)$  with  $y \in Q$  equipped with the order  $(p, q) \leq (p', q')$  if  $p \leq p'$  and  $q = q'$ ).

Indecomposable, right-indecomposable, left-indecomposable, indivisible:

An order type  $\alpha$  is *indecomposable* if  $\alpha = \beta + \gamma$  implies  $\alpha \leq \beta$  or  $\alpha \leq \gamma$ ; it is *right-indecomposable* if  $\alpha = \beta + \gamma$  with  $\gamma \neq 0$  implies  $\alpha \leq \gamma$ ; it is *strictly right-indecomposable* if  $\alpha = \beta + \gamma$  with  $\gamma \neq 0$  implies  $\beta < \alpha \leq \gamma$ . The *left-indecomposability* and the *strict left-indecomposability* are defined in the same way. An order type  $\alpha$  is *indivisible* if for every partition of a chain  $A$  of order type  $\alpha$  into  $B \cup C$ , then either  $A \leq B$  or  $A \leq C$ .

Join and meet:

The *join* or *supremum* of a subset  $X$  of a poset  $P$  is the least upper-bound of  $X$  and is denoted  $\bigvee X$ . If  $X$  is made of  $x$  and  $y$  it is denoted  $x \vee y$ . The *meet* or *infimum* of  $X$  is denoted  $\bigwedge X$ . Similarly,  $x \wedge y$  is the meet of  $x$  and  $y$ .

Compact element:

An element  $x$  of a poset  $P$  is *compact* if  $x \leq \bigvee X$  implies  $x \leq \bigvee X'$  for some finite subset  $X'$  of  $X$ .

Join-semilattice:

A *join-semilattice* is a poset  $P$  such that arbitrary elements  $x, y$  have a join.

Join-preserving:

Let  $P$  and  $Q$  be two join-semilattices. A map  $f : P \rightarrow Q$  is *join-preserving* if  $f(x \vee y) = f(x) \vee f(y)$  for every  $x, y \in P$ .

Independent set:



A subset  $X$  of a join-semilattice  $P$  is *independent* if  $x \notin \bigvee F$  for every  $x \in X$  and every non empty finite subset  $F$  of  $X \setminus \{x\}$ .

Join-irreducible, join-prime,  $\mathbb{J}_{irr}(P)$ ,  $\mathbb{J}_{pri}(P)$ :

An element  $x$  of a join-semilattice  $P$  is *join-irreducible* if it is distinct from the least element (if any) and if  $x = a \vee b$  implies  $x = a$  or  $x = b$ . We denote  $\mathbb{J}_{irr}(P)$  the set of join-irreducible elements of  $P$ . An element  $x \in P$  is *join-prime*, if it is distinct from the least element (if any) and if  $x \leq a \vee b$  implies  $x \leq a$  or  $x \leq b$ . This amounts to the fact that  $P \setminus \uparrow x$  is an ideal. We denote  $\mathbb{J}_{pri}(P)$ , the set of join-prime members of  $P$ . We have  $\mathbb{J}_{pri}(P) \subseteq \mathbb{J}_{irr}(P)$ .

Lattice, complete lattice:

A lattice is a poset  $P$  in which every pair of elements has a join and a meet. If every subset has a join and a meet,  $P$  is a complete lattice.

Algebraic lattice:

An *algebraic lattice* is a complete lattice in which every element is a join of compact elements.

Completely meet-irreducible,  $\Delta(L)$ :

Let  $L$  be a complete lattice. For  $x \in L$ , set  $x^+ := \bigwedge \{y \in L : x < y\}$ . An element  $x \in L$  is *completely meet-irreducible* if  $x = \bigwedge X$  implies  $x \in X$ , or -equivalently-  $x \neq x^+$ . We denote  $\Delta(L)$  the set of completely meet-irreducible members of  $L$ .

Sierpinskization,  $\Omega(\alpha)$ :

A *sierpinskization* of a countable order-type  $\alpha$  can be obtained by intersecting the natural order on the set  $\mathbb{N}$  of positive integers with a linear order of  $\alpha$ . Sierpinski-sations given by a bijective map  $\psi : \omega \rightarrow \omega\alpha$  such that  $\varphi^{-1}$  is order-preserving on each component  $\omega \cdot \{i\}$  of  $\omega\alpha$  are all embeddable in each other, and for this reason denoted by the same symbol  $\Omega(\alpha)$ .

$L_\alpha, \mathbb{J}, \mathbb{A}, \mathbb{L}, \mathbb{J}_{-\alpha}, \mathbb{A}_{-\alpha}, \mathbb{L}_{-\alpha}$ :

Let  $\alpha$  be a chain, we denote by  $L_\alpha := 1 + (1 \oplus \alpha) + 1$  the lattice made of the direct sum of the one-element chain 1 and the chain  $\alpha$ , with top and bottom added. We denote  $\mathbb{J}$  the class of join-semilattices having a least element,  $\mathbb{A}$  the class of algebraic lattices,  $\mathbb{L}$  the collection of  $L \in \mathbb{A}$  such that  $L$  contains no join-semilattice isomorphic to  $L_{\omega+1}$  or to  $L_{\omega^*}$ . We denote  $\mathbb{J}_{-\alpha}$  (resp.  $\mathbb{A}_{-\alpha}, \mathbb{L}_{-\alpha}$ ) the collection of  $L \in \mathbb{J}$  (resp.  $L \in \mathbb{A}, L \in \mathbb{L}$ ) such that  $L$  contains no chain of type  $I(\alpha)$ .

$K(L), \mathbb{K}$ :

Let  $L$  be an algebraic lattice, we denote  $K(L)$  the set of compact elements of  $L$ . We denote by  $\mathbb{K}$  the class of order types  $\alpha$  such that  $L \in \mathbb{L}_{-\alpha}$  whenever  $K(L)$  contains no chain of type  $1 + \alpha$  and no subset isomorphic to  $[\omega]^{<\omega}$ .



## Bibliography

- [1] G.Birkhoff, Lattice Theory, A.M.S. Coll. Pub. Vol. XXV. Third Ed., 1967.
- [2] R.Bonnet, Stratifications et extensions des genres de chaînes dénombrables, Comptes Rendus Acad. Sc.Paris, 269, Série A, (1969), 880-882.
- [3] R Bonnet, M Pouzet, Extensions et stratifications d'ensembles dispersés, Comptes Rendus Acad. Sc.Paris, 268, Série A, (1969),1512-1515.
- [4] R.Bonnet and M.Pouzet, Linear extention of ordered sets, in Ordered Sets. (I.Rival) ed. Reidel, ASI 83, (1982), 125-170.
- [5] G.Cantor, Beitrage Zur Begrundung der transfiniten Menenlehre, Math. Ann. 49 (1897), 207-246.
- [6] I.Chakir, Chaînes d'idéaux et dimension algébrique des treillis distributifs, Thèse de doctorat, Université Claude-Bernard(Lyon1) 18 décembre 1992, n 1052.
- [7] I.Chakir, M.Pouzet, The length of chains in distributive lattices, Notices of the A.M.S., 92 T-06-118, 502-503.
- [8] I.Chakir, M.Pouzet, Infinite independent sets in distributive lattices, Algebra Universalis **53**(2) 2005, 211-225.
- [9] I. Chakir, M. Pouzet, A characterization of well-founded algebraic lattices, submitted to Order, under revision.
- [10] I.Chakir, M.Pouzet, On the length of chains in algebraic modular lattices, Order, to appear.
- [11] D.Duffus, M.Pouzet, and I.Rival, Complete ordered sets with no infinite antichains, Discrete Math. 35 (1981), 39-52.
- [12] P. Erdős, A. Tarski, On families of mutually exclusive sets, Annals of Math. 44 (1943) 315-329.
- [13] R. Fraïssé. *Theory of relations*. North-Holland Publishing Co., Amsterdam, 2000.
- [14] F.Galvin, E. C. Milner, M.Pouzet, Cardinal representations for closures and preclosures, Trans. Amer. Math. Soc., 328, (1991), 667-693.
- [15] G.Grätzer, General Lattice Theory, Birkhäuser, Stuttgart, 1998.
- [16] F. Hausdorff, Grundzge einer Theorie der Geordnete Mengen, Math. Ann., 65 (1908), 435-505.
- [17] G. Higman, Ordering by divisibility in abstract algebras, Proc. London. Math. Soc. 2 (3), (1952), 326-336.
- [18] N.Hindman, Finite sums from sequences within cells of a partition of  $\mathbb{N}$ . J. Combinatorial Theory Ser. A 17 (1974), 1-11.
- [19] Hofmann, K. H., M. Mislove and A. R. Stralka, The pontryagin duality of compact 0-dimensional semilattices and its applications, Lecture Note in Mathematics 396 (1974), Springer-Verlag.
- [20] D. H. J. de Jongh and R. Parikh. Well-partial orderings and hierarchies. *Nederl. Akad. Wetensch. Proc. Ser. Math.*, 39(3):195-207, 1977.
- [21] R. Laver, On Fraïssé's order type conjecture, Annals of Math., 93 (1971), 89-111.
- [22] R. Laver, An order type decomposition theorem, Annals of Math., 98 (1973), 96-119.
- [23] J.D. Lawson, M. Mislove, H. A. Priestley, Infinite antichains in semilattices, Order, 2 (1985), 275-290.

- [24] J.D. Lawson, M. Mislove, H. A. Priestley, Infinite antichains and duality theories, *Houston Journal of Mathematics*, Volume 14, No. 3, (1988), 423-441.
- [25] J.D. Lawson, M. Mislove, H. A. Priestley, Ordered sets with no infinite antichains, *Discrete Mathematics*, 63 (1987), 225-230.
- [26] E. C. Milner and M. Pouzet, On the Cofinality of Partially Ordered Sets, (I.Rival) ed. Reidel ASI 83, (1982), 279-297.
- [27] E. C. Milner, M. Pouzet, A decomposition theorem for closure systems having no infinite independent set, in *Combinatorics, Paul Erdős is Eighty (Volume 1)*, Keszthely (Hungary), 1993, pp. 277-299, Bolyai Society Math. Studies.
- [28] M. Mislove, When are order scattered and topologically scattered the same? *Annals of Discrete Math.* 23 (1984), 61-80.
- [29] C. St. J. A. Nash-Williams, On Well-Quasi-Ordering Finite Trees, *Proc. Cambridge Philos. Soc.*, 59 (1963), 833-835.
- [30] M. Pouzet, Condition de chaîne en théorie des relations, *Israël J. Math. (Ser. 1-2)* 30 (1978) 65-84.
- [31] M. Pouzet, *Sur la théorie des relations*. Thèse d'état, Université Claude-Bernard, Lyon 1, pp. 78-85, 1978.
- [32] M. Pouzet, M. Sobrani, Sandwiches of ages, *Annals of Pure and Applied logic*, 108 (2001), 299-330
- [33] M. Pouzet, M. Sobrani, Ordinal invariants of an age. Report . August 2002, Université Claude-Bernard (Lyon 1).
- [34] M. Pouzet and N. Zaguia, Ordered sets with no chains of ideals of a given type, *Order*, 1 (1984), 159-172.
- [35] H. A. Priestley, ordered sets and duality for distributive lattices, *Annals of Discrete Math.*, 23, (1984), 39-60.
- [36] R. Rado, Partial well-ordering of a set of vectors, *Mathematika*, 1 (1954), 89-95.
- [37] F. P. Ramsey, On a problem of formal logic, *Proc. London Math. Soc.*, 30, (1930), 264-286.
- [38] J. G. Rosenstein, *Linear ordering*, Academic Press, 1982.
- [39] S. Shelah, Independence of strong partition relation for small cardinals, and the free subset problem, *The Journal of Symbolic Logic*, 45, (1980), 505-509.
- [40] M. Sobrani, *Sur les âges de relations et quelques aspects homologiques des constructions D+M*. Thèse de doctorat d'état, Université S.M. Ben Abdallah-Fez, Fez, Janvier 2002.
- [41] W.T. Trotter. *Combinatorics and Partially Ordered Sets: Dimension Theory*, The Johns Hopkins University Press, Baltimore, MD, 1992.

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